

FLATNESS OF GENERIC POISSON PAIRS IN ODD DIMENSION

FRANCISCO-JAVIER TURIEL

ABSTRACT. Given a $(m-2)$ -form ω and a volume form Ω on a m -manifold one defines a bi-vector Λ by setting $\Lambda(\alpha, \beta) = \frac{\alpha \wedge \beta \wedge \omega}{\Omega}$ for any 1-forms α, β . In this way, locally, a Poisson pair, or bi-Hamiltonian structure, (Λ, Λ_1) is always represented by a couple of $(m-2)$ -forms ω, ω_1 and a volume form Ω . Here one shows that, for $m \geq 5$ and odd and (Λ, Λ_1) generic, (Λ, Λ_1) is flat if and only if there exists a 1-form λ such that $d\omega = \lambda \wedge \omega$ and $d\omega_1 = \lambda \wedge \omega_1$.

Moreover, we use this result for constructing several examples of linear or Lie Poisson pairs that are generic and non-flat.

MSC: 37K10, 53D17, 53A60

1. INTRODUCTION

Henceforth differentiable means C^∞ in the real case and holomorphic in the complex one. Manifolds, real or complex, and objects on them are assumed to be differentiable unless another thing is stated.

Poisson pairs or bi-Hamiltonian structures, introduced by F. Magri [6] and I. Gelfand and I. Dorfman [1], are a powerful tool for integrating many equations from Physics. At the same time the study of their geometric properties, regardless other aspects, gives rise to several interesting problems, global [8, 9] and mostly local, for instance the theory of Veronese webs, notion introduced by I. Gelfand and I. Zakharevich in codimension one and later on extended to any codimension by other authors [2, 3, 7, 11, 12, 14].

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Often, for applications, it is important to determine *in a practical way* whether generic Poisson pairs are flat or not (see section 2 for definitions.)

In even dimension the problem is solved since a local classification is known [10], and flatness is equivalent to say that the Poisson pair defines a G -structure. On the contrary in odd dimension no general and practical criterion of flatness is known, except for dimension three where a simple explicit obstruction has been constructed by A. Izosimov [4]. In this work one gives a such criterion by making use of the differential forms.

More exactly, given a $(m - 2)$ -form ω and a volume form Ω on a m -manifold one defines a bi-vector Λ by setting $\Lambda(\alpha, \beta) = \frac{\alpha \wedge \beta \wedge \omega}{\Omega}$ for any 1-forms α, β . In this way, locally, a Poisson pair (Λ, Λ_1) is always represented by a couple of $(m - 2)$ -forms ω, ω_1 and a volume form Ω . Here one shows that, for $m \geq 5$ and odd and (Λ, Λ_1) generic, (Λ, Λ_1) is flat if and only if there exists a 1-form λ such that $d\omega = \lambda \wedge \omega$ and $d\omega_1 = \lambda \wedge \omega_1$.

In a second time we apply this result to generic, linear or Lie, Poisson pairs (see section 5 for definitions.) It is well known the difficulty for constructing this kind of Poisson pairs (for a non-trivial example of linear Poisson pair in dimension five see [5].) Our result allows to construct, in any dimension ≥ 5 , examples of these pairs which are non flat; for instance on the dual space of some Lie algebra of truncated polynomial fields in one variable (examples 2 and 9.)

Moreover one shows that if the dual space of a non-unimodular Lie algebra of dimension ≥ 5 supports a generic linear Poisson pair, then it supports a non-flat generic linear Poisson pair too. This result applies to some semi-direct product of the affine algebra and an ideal of dimension one (example 8.)

On the other hand from a Lie algebra one constructs a second one, called secondary, whose dual space support a generic non-flat linear Poisson pair provided that some minor conditions hold; just it is the case of the special affine algebra (example 7.)

In section 8 many examples of generic non-flat Lie Poisson pairs are given, one of them containing the affine algebra as subalgebra (see remark 6.)

From our examples follows that for the linear Lie algebra, and therefore for any Lie algebra \mathcal{A} , there exist two other Lie algebras \mathcal{B} and \mathcal{B}' , which contain \mathcal{A} as subalgebra, such that \mathcal{B}^* support a generic non-flat linear Poisson pair and \mathcal{B}'^* a generic non-flat Lie Poisson pair. Thus a natural question arises: to determine the minimal dimension of such Lie algebras and describe the structural relation among \mathcal{A} , \mathcal{B} and \mathcal{B}' .

Finally, sections 9 and 10 include a completely description of generic and non-flat linear Poisson pair or Lie Poisson pair, respectively, in dimension three.

2. PRELIMINARIES

Let V be a real or complex vector space of dimension m and $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Given a volume form Ω on V , to each $\omega \in \Lambda^{m-2}V^*$ one may associated a bi-vector Λ through the formula

$$\Lambda(\alpha, \beta) = \frac{\alpha \wedge \beta \wedge \omega}{\Omega}$$

where $\alpha, \beta \in V^*$ and the quotient means the scalar a such that $\alpha \wedge \beta \wedge \omega = a\Omega$; in this way an isomorphism between $\Lambda^{m-2}V^*$ and Λ^2V is defined. Thus a bi-vector Λ on V can be represented by a couple of forms (ω, Ω) where $\omega \in \Lambda^{m-2}V^*$ and $\Omega \in \Lambda^mV^* - \{0\}$. Of course (ω, Ω) and (ω', Ω') represent the same bi-vector if and only if $\omega' = b\omega$ and $\Omega' = b\Omega$ for some $b \in \mathbb{K} - \{0\}$.

Note that for any $\alpha \in V^*$ its Λ -Hamiltonian $\Lambda(\alpha, \quad)$ is just the vector v_α such that $i_{v_\alpha}\Omega = -\alpha \wedge \omega$.

Lemma 1. *Consider a bi-vector Λ represented by (ω, Ω) . If $\Omega = ae_1^* \wedge \dots \wedge e_m^*$ and $\Lambda = \sum_{1 \leq i < j \leq m} a_{ij} e_i \wedge e_j$, where $\{e_1, \dots, e_m\}$ is a basis of V , then*

$$\omega = \sum_{1 \leq i < j \leq m} (-1)^{i+j-1} a a_{ij} e_1^* \wedge \dots \wedge \widehat{e_i^*} \wedge \dots \wedge \widehat{e_j^*} \wedge \dots \wedge e_m^*$$

(as usual terms under hat are deleted.)

Recall that a bi-vector can be described by a r -form, whose kernel equals the image of the bi-vector, and a 2-form, whose restriction to the kernel of the r -form is symplectic; indeed, identify the bi-vector to the dual bi-vector of the restriction of the 2-form.

Lemma 2. *On V consider 1-forms $\alpha_1, \dots, \alpha_r$ and a 2-form β such that $\alpha_1 \wedge \dots \wedge \alpha_r \wedge \beta^k \neq 0$ where $m = 2k + r$. Then $\alpha_1 \wedge \dots \wedge \alpha_r, \beta$ describe a bi-vector that is represented, as well, by $(k\alpha_1 \wedge \dots \wedge \alpha_r \wedge \beta^{k-1}, \alpha_1 \wedge \dots \wedge \alpha_r \wedge \beta^k)$.*

Consider a couple of bi-vectors (Λ, Λ_1) on V . By definition the rank of (Λ, Λ_1) is the maximum of ranks of $(1-t)\Lambda + t\Lambda_1$, $t \in \mathbb{K}$. Note that $rank(\Lambda, \Lambda_1)$ equals $rank((1-t)\Lambda + t\Lambda_1)$ for any $t \in \mathbb{K}$ except a finite number of scalars. Therefore, by considering $\Lambda' = (1-a)\Lambda + a\Lambda_1$ and $\Lambda'_1 = (1-a_1)\Lambda + a_1\Lambda_1$ for suitable $a \neq a_1$, one may assume $rank\Lambda = rank\Lambda_1 = rank(\Lambda, \Lambda_1)$ if necessary.

The classification of couples (Λ, Λ_1) is due to Gelfand and Zakharevich [3, 12]; just pointing out that (Λ, Λ_1) is the product of $m - rank(\Lambda, \Lambda_1)$ Kronecker blocks and, perhaps, a symplectic factor [this last one only if there exists $b \in \mathbb{C}$ such that $rank((1-b)\Lambda + b\Lambda_1) < rank(\Lambda, \Lambda_1)$ or $rank(\Lambda - \Lambda_1) < rank(\Lambda, \Lambda_1)$.]

Let M be a real or complex manifold of dimension m and (Λ, Λ_1) a couple of bi-vectors on it. One will say that at a point p the couple (Λ, Λ_1) is:

- (1) *flat* if, in some coordinates around p , Λ and Λ_1 can be simultaneously written with constant coefficients.
- (2) a *G-structure* if there exists an open neighborhood A of p such that, for any $q \in A$, the algebraic couples $(\Lambda, \Lambda_1)(q)$ on $T_q M$ and $(\Lambda, \Lambda_1)(p)$ on $T_p M$ are isomorphic.
- (3) a *Poisson pair or a bi-Hamiltonian structure* if, on some open neighborhood of p , Λ , Λ_1 and $\Lambda + \Lambda_1$ are Poisson (structures); in this case $a\Lambda + b\Lambda_1$ is Poisson for any $a, b \in \mathbb{K}$.

When the properties above hold at every point of M , (Λ, Λ_1) is called flat, a *G-structure* or a Poisson pair respectively.

Note that (1) implies (2) and (3).

One will say that $(\omega, \omega_1, \Omega)$ represents (Λ, Λ_1) , when $(\omega, \Omega)(q)$ and $(\omega_1, \Omega)(q)$ represent $\Lambda(q)$ and $\Lambda_1(q)$ respectively for every $q \in M$. If $(\omega', \omega'_1, \Omega')$ is another representative of (Λ, Λ_1) then $(\omega', \omega'_1, \Omega') = f(\omega, \omega_1, \Omega)$, that is $\omega' = f\omega$, $\omega'_1 = f\omega_1$ and $\Omega' = f\Omega$, where f is a function without zeros. The existence of a representative of (Λ, Λ_1) on M only depends on the existence of a volume form. Since our problem is local, we may suppose that it is the case without loss of generality.

Proposition 1. *Let (Λ, Λ_1) be a Poisson pair on M represented by $(\omega, \omega_1, \Omega)$. If (Λ, Λ_1) is flat at p then, about this point, there is a closed 1-form λ such that $d\omega = \lambda \wedge \omega$ and $d\omega_1 = \lambda \wedge \omega_1$.*

Moreover if a Poisson structure $\tilde{\Lambda}$ represented by $(\tilde{\omega}, \tilde{\Omega})$ is flat on a neighborhood of p , then around this point there exists a function \tilde{g} with no zeros such that $\tilde{g}\tilde{\omega}$ is closed.

Proof. By flatness, and always around p , there exists a representative $(\omega', \omega'_1, \Omega')$ of (Λ, Λ_1) such that $d\omega' = d\omega'_1 = 0$. On the other hand $(\omega, \omega_1, \Omega) = h(\omega', \omega'_1, \Omega')$ for some function h with no zeros. Moreover one can assume $h = \pm e^g$. Then $d\omega = dg \wedge \omega$ and $d\omega_1 = dg \wedge \omega_1$.

The second part of proposition 1 is obvious □

The proposition above provides a necessary condition for flatness that in some cases, as one shows in the next two sections, is sufficient too.

3. THE GENERIC CASE IN ODD DIMENSION

In this section $m = 2n - 1 \geq 3$, so $n \geq 2$. Let (Λ, Λ_1) a couple of Poisson structures. Suppose (Λ, Λ_1) *generic* (at each point.) This is equivalent to assume that $(a\Lambda + b\Lambda_1)^{n-1}$ has no zeros for any $(a, b) \in \mathbb{C}^2 - \{0\}$; so its algebraic model has just a Kronecker block and no symplectic factor, and (Λ, Λ_1) defines a G -structure. Moreover if a 1-form τ is Casimir for both Λ and Λ_1 , that is $\Lambda(\tau, \cdot) = \Lambda_1(\tau, \cdot) = 0$, then $\tau = 0$. Thus if $(\omega, \omega_1, \Omega)$ represents (Λ, Λ_1) and $\tau \wedge \omega = \tau \wedge \omega_1 = 0$ then $\tau = 0$.

Therefore if there exists λ such that $d\omega = \lambda \wedge \omega$ and $d\omega_1 = \lambda \wedge \omega_1$, it is unique. Besides, if $(\omega', \omega'_1, \Omega')$ represents (Λ, Λ_1) as well, then we can assume $(\omega', \omega'_1, \Omega') = \pm e^g(\omega, \omega_1, \Omega)$. So $d\omega' = (\lambda + dg) \wedge \omega'$ and $d\omega'_1 = (\lambda + dg) \wedge \omega'_1$. This shows that the existence of this kind of 1-form does not depend on the representative while its exterior derivative is intrinsic. Consequently $d\lambda$ can be constructed even if representatives are local only.

In general there is no reason for the existence of a such λ except in dimension three. More exactly:

Proposition 2. *Let (Λ, Λ_1) be a couple of Poisson structures and $(\omega, \omega_1, \Omega)$ a representative. Assume $m = 3$ and (Λ, Λ_1) generic. Then there exists a 1-form λ such that $d\omega = \lambda \wedge \omega$ and $d\omega_1 = \lambda \wedge \omega_1$ if and only if (Λ, Λ_1) is a Poisson pair; in this case $\text{Im}\Lambda \cap \text{Im}\Lambda_1 \subset \text{Ker}d\lambda$.*

Moreover $d\lambda = 0$ if and only if (Λ, Λ_1) is flat.

Proof. In dimension three a bi-vector Λ' represented by (ω', Ω') is Poisson if and only if ω' is completely integrable, that is $\omega' \wedge d\omega' = 0$. Besides if $\Lambda'(q) \neq 0$ then $\text{Im}\Lambda'(q) = \text{Ker}\omega'(q)$.

In our case as $\omega \wedge d\omega = 0$ and the problem is local, there exists a function φ without zeros such that $d(\varphi\omega) = 0$. Thus changing of representative allows us to suppose ω closed.

Since $Im\Lambda$ and $Im\Lambda_1$ are in general position about any point, one may choose coordinates (x_1, x_2, x_3) such that $\omega = dx_1$ and $\omega_1 = e^h dx_2$. Moreover by modifying the third coordinate if necessary one can suppose $\Omega = dx_1 \wedge dx_2 \wedge dx_3$. The couple (Λ, Λ_1) is a Poisson pair if and only if $\Lambda + \Lambda_1$ is Poisson, that is if and only if $(\omega + \omega_1) \wedge d(\omega + \omega_1) = 0$ (note that $(\omega + \omega_1, \Omega)$ represents $\Lambda + \Lambda_1$), which is equivalent to say that $h = h(x_1, x_2)$.

On the other hand as $d\omega = 0$ the 1-form λ , if any, has to be equal gdx_1 , which implies that $d\omega_1 = e^h dh \wedge dx_2 = ge^h dx_1 \wedge dx_2$. In other words a such λ exists just when $h = h(x_1, x_2)$; in this case $\lambda = (\partial h / \partial x_1) dx_1$. That proves the first part of proposition 2.

Now suppose $d\lambda = 0$; then $\partial^2 h / \partial x_2 \partial x_1 = 0$ and $h = h_1(x_1) + h_2(x_2)$ so $e^{-h_1}\omega$ and $e^{-h_1}\omega_1$ are closed. Therefore changing of representative allows to suppose ω, ω_1 closed. Consequently there exist coordinates (y_1, y_2, y_3) such that $\omega = dy_1$ and $\omega_1 = dy_2$. By modifying the third coordinate if necessary we may suppose $\Omega = dy_1 \wedge dy_2 \wedge dy_3$; now is obvious that (Λ, Λ_1) is flat. The converse follows from proposition 1. \square

Remark 1. By proposition 2 in dimension three $d\lambda$ is an invariant of the Poisson pair (Λ, Λ_1) , which vanishes just when (Λ, Λ_1) is flat. A straightforward computation shows that, up to multiplicative constant, this invariant equals the curvature form introduced by Izosimov [4]

For odd dimension greater than or equal to five one has:

Theorem 1. *Consider a Poisson pair (Λ, Λ_1) on a manifold M of dimension $m = 2n - 1 \geq 5$ represented by $(\omega, \omega_1, \Omega)$. Assume that*

(Λ, Λ_1) is generic everywhere. Then (Λ, Λ_1) is flat if and only if there exists a 1-form λ such that $d\omega = \lambda \wedge \omega$ and $d\omega_1 = \lambda \wedge \omega_1$.

Proof. The existence of λ when (Λ, Λ_1) is flat follows from proposition 1. Conversely suppose that λ exists; for proving the flatness, which is a local question, it will be enough to show that the Veronese web associated is flat [2, 3, 11, 12].

One starts proving that $d\lambda = 0$. For every $a \in \mathbb{K}$ the Poisson structure $\Lambda + a\Lambda_1$ is represented by $(\omega + a\omega_1, \Omega)$ so, by proposition 1, (locally) there is a function g_a with no zeros such that $g_a(\omega + a\omega_1)$ is closed. Therefore $0 = (\lambda + (dg_a)/g_a) \wedge g_a(\omega + a\omega_1)$ and $\lambda + (dg_a)/g_a$ has to be a Casimir of $\Lambda + a\Lambda_1$.

This implies that $d\lambda = d(\lambda + (dg_a)/g_a)$ is divisible by any (1-form) Casimir of $\Lambda + a\Lambda_1$. Thus $d\lambda$ is divisible by every Casimir of $\Lambda + a\Lambda_1$, $a \in \mathbb{K}$. But at each point all these Casimirs span a vector space of dimension ≥ 3 . Since $d\lambda$ is a 2-form necessarily vanishes.

As it is known, given any different and non-vanishing real numbers a_1, \dots, a_n , around each point $p \in M$, there exist coordinates $(x, y) = (x_1, \dots, x_n, y_1, \dots, y_{n-1})$ and functions f_1, \dots, f_n only depending on x such that Λ is given by $(a_1 \cdots a_n) \sum_{j=1}^n a_j^{-1} f_j dx_j$ and $\sum_{j=1}^{n-1} dx_j \wedge dy_j$ while Λ_1 is given by $\alpha = \sum_{j=1}^n f_j dx_j$ and $\sum_{j=1}^{n-1} a_j dx_j \wedge dy_j$, where f_1, \dots, f_n have no zeros, $d\alpha = 0$ and $\alpha \wedge d(\alpha \circ J) = 0$ when α is regarded as a 1-form in variables $x = (x_1, \dots, x_n)$ and $J = \sum_{j=1}^n a_j (\partial/\partial x_j) \otimes dx_j$; that is $\alpha \circ J = \sum_{j=1}^n a_j f_j dx_j$ and $\alpha \circ J^{-1} = \sum_{j=1}^n a_j^{-1} f_j dx_j$ (see page 893 of [11] and section 3 of [12]). For simplifying computations we choose a_1, \dots, a_n in such a way that $a_1 \cdots a_n = 1$.

By lemma 2 the Poisson structure Λ is represented by

$$\omega' = (n-1)(\alpha \circ J^{-1}) \wedge \left(\sum_{j=1}^{n-1} dx_j \wedge dy_j \right)^{n-2}$$

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and Λ_1 by

$$\omega'_1 = (n-1)\alpha \wedge \left(\sum_{j=1}^{n-1} a_j dx_j \wedge dy_j \right)^{n-2}$$

$$\Omega'_1 = \alpha \wedge \left(\sum_{j=1}^{n-1} a_j dx_j \wedge dy_j \right)^{n-1}.$$

Note that $\Omega' = \Omega'_1$ because $a_1 \cdots a_n = 1$. On the other hand $d\omega'_1 = 0$; therefore for the representative $(\omega', \omega'_1, \Omega')$ there exists a 1-form $\lambda' = \varphi\alpha$ such that $d\omega' = \lambda' \wedge \omega'$. But $d\lambda' = d\lambda = 0$ so $\lambda' = dg$ for some function $g = g(x)$ such that $\alpha \wedge dg = 0$, and $e^{-g}\omega'$, $e^{-g}\omega'_1$ will be closed.

Observe that $\tilde{\alpha} = e^{-g}\alpha$ is closed, $e^{-g}f_1, \dots, e^{-g}f_n$ have no zeros, $\tilde{\alpha} \wedge d(\tilde{\alpha} \circ J) = 0$ while Λ , Λ_1 are given by $\tilde{\alpha} \circ J^{-1}$, $\sum_{j=1}^{n-1} dx_j \wedge dy_j$ and $\tilde{\alpha}$, $\sum_{j=1}^{n-1} a_j dx_j \wedge dy_j$ respectively. In other words, considering $\tilde{\alpha}$ instead α and calling it α again allows to assume, without loss of generality, $\lambda' = 0$ and ω' closed.

As $\text{Ker}(\alpha \circ J^{-1}) = \text{Im}\Lambda$ is involutive $d(\alpha \circ J^{-1}) = \tau \wedge (\alpha \circ J^{-1})$ for some 1-form τ . Hence $0 = d\omega' = \tau \wedge \omega'$ since $(\sum_{j=1}^{n-1} dx_j \wedge dy_j)^{n-2}$ is closed. Therefore τ is a Casimir of Λ and $\tau = h\alpha \circ J^{-1}$ for some function h . But in this case $\tau \wedge (\alpha \circ J^{-1}) = 0$ and $\alpha \circ J^{-1}$ will be closed. In other words

$$0 = d(\alpha \circ J^{-1}) = \sum_{1 \leq k < j \leq n} [a_j^{-1}(\partial f_j / \partial x_k) - a_k^{-1}(\partial f_k / \partial x_j)] dx_k \wedge dx_j$$

so $a_j^{-1}(\partial f_j / \partial x_k) - a_k^{-1}(\partial f_k / \partial x_j) = 0$. Since $(\partial f_j / \partial x_k) = (\partial f_k / \partial x_j)$, necessarily $(\partial f_j / \partial x_k) = 0$ if $j \neq k$. Thus $f_j = f_j(x_j)$, $j = 1, \dots, n$.

Recall that the web associated to (Λ, Λ_1) is completely determined by J and α regarded on the (local) quotient manifold N of M by the foliation $\bigcap_{t \in \mathbb{K}} \text{Im}(\Lambda + t\Lambda_1)$ (see [11, 12] again;) of course (x_1, \dots, x_n) can be regarded too as coordinates on N around q , where q is the projection on N of point p . Now consider functions $\tilde{x}_1, \dots, \tilde{x}_n$ such

that $d\tilde{x}_j = f_j(x_j)dx_j$, $j = 1, \dots, n$; then $(\tilde{x}_1, \dots, \tilde{x}_n)$ are coordinates on N about q , $\alpha = d\tilde{x}_1 + \dots + d\tilde{x}_n$ and $J = \sum_{j=1}^n a_j(\partial/\partial\tilde{x}_j) \otimes d\tilde{x}_j$, which shows the flatness of the web associated to (Λ, Λ_1) . \square

Example 1. On \mathbb{K}^5 with coordinates $x = (x_1, x_2, x_3, x_4, x_5)$ consider the bi-vectors

$$\Lambda = \left(x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_3} \right) \wedge \frac{\partial}{\partial x_4} + \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} \right) \wedge \frac{\partial}{\partial x_5}$$

$$\Lambda_1 = x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_4} + \left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) \wedge \frac{\partial}{\partial x_5}.$$

First note that $\Lambda + t\Lambda_1 = X_t \wedge (\partial/\partial x_4) + Y_t \wedge (\partial/\partial x_5)$ where X_t , Y_t , $(\partial/\partial x_4)$ and $(\partial/\partial x_5)$ commute among them, which implies that every $\Lambda + t\Lambda_1$ is a Poisson structure and (Λ, Λ_1) a Poisson pair (in fact a Lie Poisson pair, see section 5.)

Let A be the complement of the union of hyperplanes $x_k = 0$, $k = 1, 2, 3$, and in addition to these $x_2 = (a_j + 1)x_1$, $j = 1, 2$, where a_1, a_2 are the roots of $t^2 + t + 1 = 0$, if $\mathbb{K} = \mathbb{C}$. Then (Λ, Λ_1) is generic on A .

Set $\Omega = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5$. Then Λ is represented by

$$\begin{aligned} \omega = & (-x_3 dx_1 \wedge dx_2 + x_2 dx_1 \wedge dx_3 - x_1 dx_2 \wedge dx_3) \wedge dx_4 \\ & + (x_3 dx_1 \wedge dx_2 + x_2 dx_2 \wedge dx_3) \wedge dx_5 \end{aligned}$$

and Λ_1 by $\omega_1 = (x_2 dx_1 - x_1 dx_2) \wedge dx_3 \wedge dx_4 - x_1 dx_1 \wedge dx_3 \wedge dx_5$.

Moreover

$$d\omega = -3dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5$$

and

$$d\omega_1 = -2dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 = (2x_1^{-1}dx_1) \wedge \omega_1.$$

As dx_3 is a Casimir of Λ_1 , if $d\omega_1 = \lambda \wedge \omega_1$ then $\lambda = 2x_1^{-1}dx_1 + f dx_3$.

On the other hand if we assume $d\omega = \lambda \wedge \omega$ then $\Lambda(\lambda, \cdot) = (\partial/\partial x_4) + 3(\partial/\partial x_5)$ since the contraction of $-\Lambda(\lambda, \cdot)$ and Ω equals $\lambda \wedge \omega = d\omega$. But at the same time

$$\Lambda(\lambda, \cdot) = \frac{2x_2}{x_1} \frac{\partial}{\partial x_4} + 2 \frac{\partial}{\partial x_5} + x_3 f \left(\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5} \right)$$

which implies that $(2x_1^{-1}x_2 + x_3f)$ and x_3f have to be constant, *contradiction*. In short (Λ, Λ_1) is not flat at any point of A .

A practical criterion for local flatness is the following:

Lemma 3. *Let M , (Λ, Λ_1) and $(\omega, \omega_1, \Omega)$ be as in theorem 1. On some neighborhood of a point p of M consider a vector field X tangent to $\text{Ker} d\omega$ and a 1-form α Casimir of Λ_1 , both of them non-vanishing at p . Assume $d\omega(p) \neq 0$ and $d\omega_1 = 0$ on this neighborhood. Then (Λ, Λ_1) is flat at p if and only if $\Lambda(\alpha, \cdot)$, about p , is functionally proportional to X .*

Proof. In this proof objects will be considered on a suitable neighborhood of p . First observe that $\Lambda(\alpha, \cdot)$ does not vanishes anywhere because no common Casimir of Λ and Λ_1 other than zero exists. Recall that $i_{\Lambda(\alpha, \cdot)}\Omega = -\alpha \wedge \omega$. Thus there is a function f such that $d\omega = f\alpha \wedge \omega$ if and only if $\Lambda(\alpha, \cdot)$ and X are functionally proportional. When $\Lambda(\alpha, \cdot)$ and X are proportional it suffices setting $\lambda = f\alpha$.

Conversely if λ exists necessarily $\lambda = f\alpha$ since $d\omega_1 = 0$, so $d\omega = \lambda \wedge \omega = f\alpha \wedge \omega$ and $\Lambda(\alpha, \cdot)$ and X are proportional. \square

4. OTHER CASES

The foregoing results extend to any analytic Poisson pair (Λ, Λ_1) on a m -manifold M provided that $\text{rank}(\Lambda, \Lambda_1) = m, m-1$. Indeed, if m is even and $\text{rank}(\Lambda, \Lambda_1) = m$, then (Λ, Λ_1) is flat at a point $p \in M$ if and only if it defines a G -structure. This is a straightforward consequence of the classification of pairs of compatible symplectic forms [10] since,

up to linear combination, one may suppose symplectic Λ and Λ_1 ; in this case analyticity is not need. On the other hand:

Theorem 2. *On a real analytic or complex manifold M , of odd dimension m , consider a point $\in M$ and a Poisson pair (Λ, Λ_1) , represented by $(\omega, \omega_1, \Omega)$, whose rank equals $m - 1$. Then (Λ, Λ_1) is flat at p if and only if (Λ, Λ_1) defines a G -structure at p and there is a closed 1-form λ , about p , such that $d\omega = \lambda \wedge \omega$ and $d\omega_1 = \lambda \wedge \omega_1$.*

Proof. As the problem is local, up to linear combination, one may suppose $\text{rank}\Lambda = \text{rank}\Lambda_1 = \text{rank}(\Lambda, \Lambda_1) = m - 1$. Obviously the conditions of the theorem are necessary (see proposition 1.) Conversely, since (Λ, Λ_1) defines a G -structure at p , from the product theorem for Poisson pairs [13] follows that, always about p , (M, Λ, Λ_1) splits into a product of two Poisson pairs $(M', \Lambda', \Lambda'_1) \times (M'', \Lambda'', \Lambda''_1)$, $p = (p', p'')$, the first one Kronecker with a single block, so generic, and symplectic the second one.

Moreover (Λ'', Λ''_1) defines a G -structure at p'' because (Λ, Λ_1) does at p ; therefore it is flat at p'' and one can choose a representative $(\omega'', \omega''_1, \Omega'')$ with $d\omega'' = d\omega''_1 = 0$.

If $\dim M' = 1$ the proof is finished since $\Lambda' = \Lambda'_1 = 0$; therefore assume $\dim M' \geq 3$. Let $(\omega', \omega'_1, \Omega')$ be a representative of (Λ', Λ'_1) . Then with the obvious identification (by means of the pull-back by the canonical projections forms on M' or M'' can be regarded as forms on M)

$$(\omega' \wedge \Omega'' + \Omega' \wedge \omega'', \omega'_1 \wedge \Omega'' + \Omega' \wedge \omega''_1, \Omega' \wedge \Omega'')$$

represents (Λ, Λ_1) .

Note that the existence of a closed form as in theorem 2 happens for any representative because all of them are functionally proportional. Let us denote by $\tilde{\lambda}$ that corresponding to the representative above and λ' its restriction to $M' \times \{p''\}$, identified to M' in the obvious way. Then

λ' is closed, $d\omega' = \lambda' \wedge \omega'$ and $d\omega'_1 = \lambda' \wedge \omega'_1$. Therefore by proposition 2 and theorem 1 (Λ', Λ'_1) is flat at p' , which allows to conclude the flatness of (Λ, Λ_1) at p . \square

Remark 2. In the real case the analyticity is needed for applying the product theorem; nevertheless an unpublished results by the author states that if (Λ, Λ_1) defines a G -structure and $\text{rank}(\Lambda, \Lambda_1) = \dim M - 1$ then the product theorem holds in the C^∞ -category too.

If $\dim M' = 3$ then there always exists λ because this fact essentially depends on $(M', \Lambda', \Lambda'_1)$. By the same reason when $\dim M' \geq 5$ the condition $d\lambda = 0$ is unnecessary.

5. PAIRS ON THE DUAL SPACE OF A LIE ALGEBRA

The remainder of this work is essentially devoted to the generic case in odd dimension when Λ is linear and Λ_1 linear or constant.

Lemma 4. *Consider a couple of analytic bi-vectors (Λ, Λ_1) on a connected non-empty open set $A \subset \mathbb{K}^m$, $m = 2n - 1 \geq 3$. If $(\Lambda, \Lambda_1)(p)$ is generic for some $p \in A$, then the set of points $q \in A$ such that $(\Lambda, \Lambda_1)(q)$ is generic is open and dense.*

Proof. Fixed $s \in \mathbb{K}$ the equation $(\Lambda + s\Lambda_1)(\lambda, \quad) = 0$ is an homogeneous linear system, on $(\mathbb{K}^m)^*$, with analytic coefficients (like functions on A .) Since $(\Lambda + s\Lambda_1)^{n-1}(p) \neq 0$, at each point near p the vector subspace of its solutions has dimension one. Thus around p there is a solution $\lambda_s = \sum_{j=1}^m f_j dx_j$, with $\lambda_s(p) \neq 0$, where every f_j is a rational function of the coefficients functions of Λ and Λ_1 . Multiplying by a suitable polynomial allows to assume, without loss of generality, that f_1, \dots, f_n are polynomials in the coefficients functions of Λ and Λ_1 . Therefore λ_s is defined on A and by analyticity $(\Lambda + s\Lambda_1)(\lambda_s, \quad) = 0$ everywhere.

Consider n different scalars s_1, \dots, s_n . Then $(\lambda_{s_1} \wedge \dots \wedge \lambda_{s_n})(p) \neq 0$ and by analyticity $\lambda_{s_1} \wedge \dots \wedge \lambda_{s_n}$ does not vanish at any point of a dense open set A' .

By a similar reason each $(\Lambda + s_j \Lambda_1)^{n-1}$ does not vanish at any point of a dense open set A_j . Take any $q \in A' \cap A_1 \cap \dots \cap A_n$; then $(\Lambda + s_j \Lambda_1)^{n-1}(q) \neq 0$, $j = 1, \dots, n$, and $(\lambda_{s_1} \wedge \dots \wedge \lambda_{s_n})(q) \neq 0$, which only is possible if $(\Lambda, \Lambda_1)(q)$ is generic. \square

Remark 3. It is easily seen that lemma 4 holds in analytic connected manifolds too.

Let \mathcal{A} be a Lie algebra of finite dimension and G a connected Lie group of algebra \mathcal{A} . Then any element of $\Lambda^r \mathcal{A}$, respectively $\Lambda^r \mathcal{A}^*$, can be regarded like a left invariant r -field, respectively r -form, on G . Thus we may consider the Lie and exterior derivatives on \mathcal{A} and from the formulas on G deduce the corresponding formulas on \mathcal{A} . Here we adopt the differentiable point of view or the algebraic one depending on the convenience for working with. Recall that closed means cocycle and exact cobord. One say that $\rho \in \mathcal{A}^*$ is a *contact form* if $\dim \mathcal{A} = 2k + 1$ and $\rho \wedge (d\rho)^k$ is a volume form, while $\beta \in \Lambda^2 \mathcal{A}^*$ is called *symplectic* when $d\beta = 0$ and $\text{rank} \beta = \dim \mathcal{A}$.

Remember that on \mathcal{A}^* one defines the *Lie-Poisson structure* Λ by considering the elements of \mathcal{A} as linear functions and setting $\Lambda(a, b) = [a, b]$. In the coordinates associated to any basis of \mathcal{A}^* the coefficients of Λ are linear functions. Conversely any Poisson structure on a vector space with linear coefficients is obtained in this way.

On the other hand a 2-form $\beta \in \Lambda^2 \mathcal{A}^*$ can be seen as a Poisson structure Λ on \mathcal{A}^* with constant coefficients. Remember that (Λ, Λ_1) is compatible if and only if $d\beta = 0$. In this case (Λ, Λ_1) will be named *a linear Poisson pair or a linear bi-Hamiltonian structure*.

Other option consists in considering a second Lie algebra structure $[\ , \]_1$ on \mathcal{A} and its associated Lie-Poisson structure Λ_1 on \mathcal{A}^* . Then

(Λ, Λ_1) is compatible if and only if $[\cdot, \cdot] + [\cdot, \cdot]_1$ defines a Lie algebra structure. In this case (Λ, Λ_1) will be called a *Lie Poisson pair* or a *Lie bi-Hamiltonian structure*.

In both cases, because lemma 4, when $\dim \mathcal{A} = 2n - 1 \geq 3$ one will say that (Λ, Λ_1) is *generic* if it is generic at some point (so almost everywhere), and *flat* if it is flat at every generic point.

Set $I_0 = \{a \in \mathcal{A} \mid \text{tr}[a, \cdot] = 0\}$ where $\text{tr}[a, \cdot]$ is the trace of the adjoint endomorphism $[a, \cdot]: b \in \mathcal{A} \rightarrow [a, b] \in \mathcal{A}$. Note that I_0 is an ideal, which contains the derived ideal of \mathcal{A} , and that we will call the *unimodular ideal* of \mathcal{A} . Its codimension equals zero when \mathcal{A} is unimodular and one otherwise. As Lie algebra I_0 is unimodular.

Lemma 5. *Let $\{e_1, \dots, e_m\}$ be a basis of a Lie algebra \mathcal{A} and (x_1, \dots, x_m) the coordinates on \mathcal{A}^* associated to the dual basis. Set $\Omega = dx_1 \wedge \dots \wedge dx_m$ and $X = \sum_{j=1}^m (\text{tr}[e_j, \cdot])(\partial/\partial x_j)$. Assume that (ω, Ω) represents the Lie-Poisson structure Λ of \mathcal{A}^* . Then $d\omega = i_X \Omega$.*

Therefore $d\omega = 0$ if and only if \mathcal{A} is unimodular.

Proof. Suppose that $[e_i, e_j] = \sum_{k=1}^m a_{ij}^k e_k$; then

$$\Lambda = \sum_{1 \leq i < j \leq m} \left(\sum_{k=1}^m a_{ij}^k x_k \right) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$$

and by lemma 1

$$\omega = \sum_{1 \leq i < j \leq m} (-1)^{i+j-1} \left(\sum_{k=1}^m a_{ij}^k x_k \right) dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx_m.$$

Finally a straightforward computation yields

$$\begin{aligned} d\omega &= \sum_{j=1}^m (-1)^j \left(\sum_{i=1}^m a_{ij}^i \right) dx_1 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx_m \\ &= \sum_{j=1}^m (-1)^{j-1} (\text{tr}[e_j, \cdot]) dx_1 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx_m. \end{aligned}$$

□

Remark 4. Assume $m \geq 3$ and odd. First consider a constant Poisson structure Λ_1 on \mathcal{A}^* such that (Λ, Λ_1) is compatible and generic. If \mathcal{A} is unimodular, as Λ_1 is represented by a constant $(m-2)$ -form ω_1 , by proposition 2 and theorem 1 the pair (Λ, Λ_1) is flat.

Now suppose Λ_1 given by a second Lie algebra structure $[\ , \]_1$ and (Λ, Λ_1) compatible and generic. Again by proposition 2 and theorem 1, when $[\ , \]$ and $[\ , \]_1$ are unimodular, (Λ, Λ_1) is flat. Flatness happens as well if the derived ideal of $[\ , \]$ equals \mathcal{A} ; indeed, for $a \neq 0$ small enough the derived ideal of $[\ , \] + a[\ , \]_1$ equals \mathcal{A} too; therefore $[\ , \]$ and $[\ , \] + a[\ , \]_1$ are unimodular and $(\Lambda, \Lambda + a\Lambda_1)$ flat.

A couple of 2-forms β, β_1 on a vector space V , of dimension $2n-1 \geq 3$, is named *generic* if it is generic as bi-vectors on V^* , that is if $(s\beta + t\beta_1)^{n-1} \neq 0$ for any $(s, t) \in \mathbb{C}^2 - \{0\}$; note that when $\beta_1^{n-1} \neq 0$ it suffices to check that $(\beta + t\beta_1)^{n-1} \neq 0$ for any $t \in \mathbb{C}$.

Observe that if Λ is the Lie-Poisson structure on the dual space \mathcal{B}^* of a Lie algebra \mathcal{B} , then $\Lambda(\alpha) = -d\alpha$ for each $\alpha \in \mathcal{B}^*$, where $d\alpha$ is regarded as constant bi-vector on \mathcal{B}^* . Therefore, when $\dim \mathcal{B} = 2n-1 \geq 3$, a linear (respectively Lie) Poisson pair (Λ, Λ_1) on \mathcal{B}^* , where Λ_1 is associated to $\beta \in \Lambda^2 \mathcal{B}^*$ (respectively to a second bracket $[\ , \]_1$), is generic at $\alpha \in \mathcal{B}^*$ if and only if $(d\alpha, \beta)$ (respectively $(d\alpha, d_1\alpha)$) is generic.

Example 2 (*Truncated algebra*). Let \mathcal{P} be the Lie algebra of vector fields $f \cdot (\partial/\partial u)$, on \mathbb{K} , such that f is a polynomial in u and $f(0) = 0$. Set $\mathcal{P}_m = u^m \mathcal{P}$, $m \in \mathbb{N}$, which is an ideal of \mathcal{P} of codimension m . The quotient $\mathcal{A} = \mathcal{P}/\mathcal{P}_m$ (we omit the subindex for sake of simplicity) is a Lie algebra of dimension m that we called the truncated Lie algebra (of \mathcal{P} at order m). Denote by e_k the class of $u^k(\partial/\partial u)$; then $\{e_1, \dots, e_m\}$ is a basis of \mathcal{A} and $[e_i, e_j] = (j-i)e_{i+j-1}$ if $i+j \leq m+1$ and zero otherwise.

Throughout this example one will assume $m = 2n - 1 \geq 5$. Then $de_m^* = -\sum_{j=1}^{n-1} 2(n-j)e_j^* \wedge e_{2n-j}^*$ and $de_{m-1}^* = -\sum_{j=1}^{n-1} (2(n-j)-1)e_j^* \wedge e_{2n-j-1}^*$. Moreover (de_m^*, de_{m-1}^*) is generic; this follows from the next lemma since $(de_m^*)^{n-1}$ and $(de_{m-1}^*)^{n-1}$ do not vanish.

Lemma 6. *For every $t \in \mathbb{C} - \{0\}$ one has $(e_m^* + te_{m-1}^*) \wedge (de_m^* + tde_{m-1}^*)^{n-1} \neq 0$.*

Proof. Regard \mathcal{A} like a vector subspace of the $2n$ -dimensional vector space V of basis $\{e_1, \dots, e_m, \tilde{e}\}$ and \mathcal{A}^* as a vector subspace of V^* in the obvious way ($\alpha(\tilde{e}) = 0$ for every $\alpha \in \mathcal{A}^*$). Set $\tau = -\sum_{j=1}^{n-1} 2(n-j)e_j^* \wedge e_{2n-j}^* + e_n^* \wedge \tilde{e}^*$, which is a symplectic form, and $\tau_1 = -\sum_{j=1}^{n-1} (2(n-j)-1)e_j^* \wedge e_{2n-j-1}^*$. Let J be the endomorphism given by the formula $\tau_1(v, w) = \tau(Jv, w)$. Then $J = \sum_{j=1}^{n-2} a_j e_{j+1} \otimes e_j^* + a\tilde{e} \otimes e_{n-1}^* + \sum_{k=n}^{2(n-1)} b_k e_{k+1} \otimes e_k^*$ for some non-vanishing scalars $a_j, a, b_k, j = 1, \dots, n-2, k = n, \dots, 2(n-1)$. Note that J is nilpotent so every $I + tJ, t \in \mathbb{C}$, is invertible and each $\tau + t\tau_1, t \in \mathbb{C}$, symplectic.

Moreover the vector subspace spanned by $\{e_1, \dots, e_{n-1}, \tilde{e}\}$ is Lagrangian for all symplectic form $\tau + t\tau_1, t \in \mathbb{C}$. Thus there exists a vector $v(t) = \sum_{j=1}^{n-1} c_j(t)e_j + c(t)\tilde{e}$ such that $i_{v(t)}(\tau + t\tau_1) = e_m^* + te_{m-1}^*, t \in \mathbb{C}$. It is easily checked that $c(t) \neq 0$ when $t \neq 0$; in this case $v(t) \notin \mathcal{A}$ and the restriction of $i_{v(t)}(\tau + t\tau_1)^n$ to \mathcal{A} does not vanish. But this restriction equals $n(e_m^* + te_{m-1}^*) \wedge (de_m^* + tde_{m-1}^*)^{n-1}$. \square

On \mathcal{A}^* consider coordinates (x_1, \dots, x_m) associates to the basis $\{e_1^*, \dots, e_m^*\}$ and the Lie -Poisson structure Λ . Set $\Lambda_1 \equiv de_m^*$. As (de_m^*, de_{m-1}^*) is generic, (Λ, Λ_1) is generic at $e_{m-1}^* \in \mathcal{A}^*$. Besides *it is not flat at this point*. Indeed, dx_n is a Casimir of Λ_1 and

$$\Lambda(dx_n, \quad) = (1-n)x_n \frac{\partial}{\partial x_1} + (2-n)x_{n+1} \frac{\partial}{\partial x_2} + \sum_{j=3}^{n-1} f_j \frac{\partial}{\partial x_j}.$$

In our case the vector field X of lemma 5 equals $a(\partial/\partial x_1)$ with $a \neq 0$. Since $\Lambda(dx_n, \quad)$ and $\partial/\partial x_1$ are not proportional about e_{m-1}^* ,

the non-flatness of (Λ, Λ_1) at e_{m-1}^* follows from lemma 3 (see remark below).

Remark 5. Recall that a linear vector field on a vector space V is proportional, about some point $p \in V$, to a constant vector field if and only if the associated endomorphism has rank zero or one. Let \mathcal{B} be a finite dimensional Lie algebra and Λ' the Lie-Poisson structure of \mathcal{B}^* . Consider a constant 1-form α on \mathcal{B}^* ; then there exists just a $b_\alpha \in V$ such that α is the exterior derivative of b_α regarded as a linear function on \mathcal{B}^* . On the other hand, $\Lambda'(\alpha, \cdot)$ is a linear vector field whose associated endomorphism is the dual map of $[b_\alpha, \cdot]: \mathcal{B} \rightarrow \mathcal{B}$. Thus $\Lambda'(\alpha, \cdot)$ is proportional, about some $p \in \mathcal{B}^*$, to a constant vector field if and only if $\text{rank}[b_\alpha, \cdot] \leq 1$.

Therefore if $\beta \in \Lambda^2 \mathcal{B}^*$ is a cocycle and α a Casimir of $\Lambda'' \equiv \beta$, that is $\beta(b_\alpha, \cdot) = 0$, such that $\text{rank}[b_\alpha, \cdot] \geq 2$, then the linear vector field $\Lambda'(\alpha, \cdot)$ is never proportional to a constant vector field about any element of \mathcal{B}^* . Just is the case of the foregoing example, where $\Lambda_1 \equiv de_m^*$, $\alpha = dx_n$ and $b_\alpha = e_n$.

6. OTHER RESULTS FOR THE LINEAR CASE

For non-unimodular Lie algebras, from a flat Poisson pair one may construct another one that is not flat. More exactly:

Proposition 3. *Consider a linear Poisson pair (Λ, Λ_1) on the dual space \mathcal{A}^* of a non-unimodular Lie algebra \mathcal{A} , of dimension $m = 2n - 1 \geq 5$, and an element p of \mathcal{A}^* . If (Λ, Λ_1) is generic and flat at p , then every linear Poisson pair $(\Lambda, \Lambda_1 + a\Lambda(p))$, $a \in \mathbb{K} - \{0\}$, is generic and non-flat at p .*

Therefore, given a non-unimodular Lie algebra $\tilde{\mathcal{A}}$ of odd dimension ≥ 5 , if there exist $\tilde{\alpha}, \tilde{\beta} \in \tilde{\mathcal{A}}^$ such that $(d\tilde{\alpha}, d\tilde{\beta})$ is generic, then on $\tilde{\mathcal{A}}^*$ there is a generic and non-flat linear Poisson pair.*

Proof. Clearly $(\Lambda, \Lambda_1 + a\Lambda(p))$ is generic at p . On the other hand, let X be the vector field of lemma 5 and α, α_a 1-forms on \mathcal{A}^* Casimir of Λ_1 and $\Lambda_1 + a\Lambda(p)$ respectively, both of them non-vanishing at p . As $a \neq 0$ vectors $\Lambda(\alpha, \cdot)(p)$ and $\Lambda(\alpha_a, \cdot)(p)$ are linearly independent (that follows from being generic since $\dim \mathcal{A} \geq 5$.) By lemma 3, $X(p)$ and $\Lambda(\alpha, \cdot)(p)$ are linearly dependent because (Λ, Λ_1) is flat. Therefore X and $\Lambda(\alpha_a, \cdot)$ are independent at p , so about p .

For the second assertion it is enough to remark that $(\tilde{\Lambda}, \tilde{\Lambda}_1)$, where $\tilde{\Lambda}$ is the Lie-Poisson structure on $\tilde{\mathcal{A}}^*$ and $\tilde{\Lambda}_1 \equiv d\tilde{\beta}$, is generic at $\tilde{\alpha}$; if $(\tilde{\Lambda}, \tilde{\Lambda}_1)$ is flat at $\tilde{\alpha}$ apply the first statement. \square

Example 3. Let \mathcal{A} be the Lie algebra of basis $\{e_1, \dots, e_5\}$ given by $[e_1, e_5] = e_5$, $[e_2, e_3] = e_3$, $[e_2, e_4] = -e_4$ and $[e_i, e_j] = 0$, $i < j$, otherwise (that corresponds on \mathbb{K}^3 to vector fields $e_1 = \partial/\partial z_1$, $e_2 = \partial/\partial z_2$, $e_3 = \exp(z_2)(\partial/\partial z_3)$, $e_4 = \exp(-z_2)(\partial/\partial z_3)$ and $e_5 = \exp(z_1)(\partial/\partial z_1)$). Then with respect to the coordinates associated to the dual basis $\{e_1^*, \dots, e_5^*\}$ one has

$$\Lambda = x_5 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_5} + x_3 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} - x_4 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_4}.$$

On the other hand $e_1^* \wedge e_2^* + e_3^* \wedge e_4^*$ is a 2-cocycle, so (Λ, Λ_1) , where $\Lambda_1 = (\partial/\partial x_1) \wedge (\partial/\partial x_2) + (\partial/\partial x_3) \wedge (\partial/\partial x_4)$, is a Poisson pair. It is easily checked that dx_5 is a Λ_1 -Casimir and (Λ, Λ_1) generic at $p = (0, 0, 1, 0, 1)$.

The vector field X given by lemma 5 equals $\partial/\partial x_1$ while $\Lambda(dx_5, \cdot) = -x_5(\partial/\partial x_1)$, so (Λ, Λ_1) is flat at p (lemma 3.) Nevertheless $(\Lambda, \Lambda_1 + a\Lambda(p))$, $a \in \mathbb{K} - \{0\}$, is not flat (proposition 3.) For checking this fact directly, observe that $dx_5 - adx_2 - a^2dx_4$ is a Casimir of

$$\Lambda_1 + a\Lambda(p) = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4} + a \left(\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \right)$$

and

$$\Lambda(dx_5 - adx_2 - a^2dx_4, \cdot)(p) = -\frac{\partial}{\partial x_1} - a\frac{\partial}{\partial x_3};$$

now apply lemma 3.

For unimodular algebras a more sophisticated construction is needed. Until the end of this section \mathcal{A} will denote a Lie algebra of dimension $m = 2n - 1 \geq 3$. To each element $\alpha \in \mathcal{A}^*$ such that $(d\alpha)^{n-1} \neq 0$ one associates a Lie subalgebra \mathcal{A}_α as follows:

- (1) If $\alpha \wedge (d\alpha)^{n-1} \neq 0$ then $\mathcal{A}_\alpha = 0$.
- (2) If $\alpha \wedge (d\alpha)^{n-1} = 0$ then there exists $v \in \mathcal{A}$ such that $i_v d\alpha = -\alpha$; moreover if $i_w d\alpha = -\alpha$ for another $w \in \mathcal{A}$ then $w - v \in \text{Ker} d\alpha$. By definition \mathcal{A}_α will be the 2-dimensional vector space spanned by v and $\text{Ker} d\alpha$, which is a Lie subalgebra because one has $L_v d\alpha = -d\alpha$ that implies $[v, \text{Ker} d\alpha] \subset \text{Ker} d\alpha$.

In this case v will be named a *Hamiltonian of α* .

Lemma 7. *Suppose \mathcal{A} unimodular. Consider $\alpha \in \mathcal{A}^*$ such that $(d\alpha)^{n-1} \neq 0$. Then \mathcal{A}_α is either non-abelian or zero.*

Proof. Assume $\dim \mathcal{A}_\alpha = 2$. Let v a Hamiltonian of α and u a basis of $\text{Ker} d\alpha$. Take $\tau \in \Lambda^m \mathcal{A}^* - \{0\}$. Then $i_u \tau = c(d\alpha)^{n-1}$ with $c \neq 0$.

Since \mathcal{A} is unimodular, $L_v \tau = 0$; now if $[u, v] = 0$ one has

$$0 = i_u L_v \tau = L_v (i_u \tau) = c L_v ((d\alpha)^{n-1}) = -c(n-1)(d\alpha)^{n-1} \neq 0$$

contradiction. □

For the purpose of this work, a couple $(\alpha, \beta) \in \mathcal{A}^* \times \mathcal{A}^*$ will be called *generic* if $(d\alpha, d\beta)$ is generic and $\mathcal{A}_{(s\alpha+t\beta)}$ is non-abelian or zero for all $(s, t) \in \mathbb{C}^2 - \{0\}$; note that if \mathcal{A}_β is non-abelian or zero, it suffices to check the property for every $\mathcal{A}_{(\alpha+t\beta)}$, $t \in \mathbb{C}$. Even when \mathcal{A} is a real Lie algebra, this definition is meaningful by complexifying it. On the other hand, by lemma 7, when \mathcal{A} is unimodular (α, β) is generic if $(d\alpha, d\beta)$ is generic.

The next step is to construct a new Lie algebra $\mathcal{B}_{\mathcal{A}}$, called *the secondary algebra* (of \mathcal{A}). Set $\mathcal{B}_{\mathcal{A}} = \mathcal{A} \times \mathcal{A} \times \mathbb{K}$ endowed with the bracket

$$[(v, v', s), (w, w', t)] = ([v, w], [v, w'] - [w, v'] + tv' - sw', 0).$$

This new algebra is just the extension of \mathcal{A} by \mathcal{A} , this second time regarded as an abelian ideal, by means of the adjoint representation plus more one dimension for taking into account $I: \mathcal{A} \rightarrow \mathcal{A}$. Clearly $\mathcal{B}_{\mathcal{A}}$ is not unimodular and its dimension equals $2m + 1$.

If $\{\tilde{e}_1, \dots, \tilde{e}_m\}$ is a basis of \mathcal{A} , $[\tilde{e}_i, \tilde{e}_j] = \sum_{k=1}^m c_{ij}^k \tilde{e}_k$, $i, j = 1, \dots, m$, and we set $e_r = (\tilde{e}_r, 0, 0)$, $f_r = (0, \tilde{e}_r, 0)$, $r = 1, \dots, m$ and $e = (0, 0, 1)$, then $\{e_1, \dots, e_m, f_1, \dots, f_m, e\}$ is a basis of $\mathcal{B}_{\mathcal{A}}$ and $[e_i, e_j] = \sum_{k=1}^m c_{ij}^k e_k$, $[e_i, f_j] = -[f_j, e_i] = \sum_{k=1}^m c_{ij}^k f_k$, $[f_j, e] = -[e, f_j] = f_j$, $i, j = 1, \dots, m$, while the other brackets vanish.

In coordinates $(x, y, z) = (x_1, \dots, x_m, y_1, \dots, y_m, z)$ associated to the dual basis $\{e_1^*, \dots, e_m^*, f_1^*, \dots, f_m^*, e^*\}$ the Lie-Poisson structure on $\mathcal{B}_{\mathcal{A}}^*$ writes:

$$\begin{aligned} \Lambda = & \sum_{1 \leq i < j \leq m} \sum_{k=1}^m c_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \\ & + \sum_{i,j,k=1}^m c_{ij}^k y_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_j} + \left(\sum_{k=1}^m y_k \frac{\partial}{\partial y_k} \right) \wedge \frac{\partial}{\partial z}. \end{aligned}$$

Proposition 4. *Let \mathcal{A} be a Lie algebra of dimension $m = 2n - 1 \geq 3$. If there exists a generic couple $(\alpha, \beta) \in \mathcal{A}^* \times \mathcal{A}^*$ such that $\beta \wedge (d\beta)^{n-1} \neq 0$, then on $\mathcal{B}_{\mathcal{A}}^*$ there is some non-flat generic linear Poisson pair.*

Proof. Observe that the center of \mathcal{A} is zero because it is included in $\text{Ker} d\alpha \cap \text{Ker} d\beta$, which is zero since $(d\alpha, d\beta)$ is generic. Regarding $(d\alpha, d\beta)$ as a couple of bi-vectors on \mathcal{A}^* and taking into account that Casimirs of $d\alpha + td\beta$, $t \in \mathbb{K}$, correspond to elements of $\text{Ker}(d\alpha + td\beta)$ show the existence of a polynomial curve $\gamma(t) = \sum_{j=0}^n t^j a_j$ in \mathcal{A} , with a_0, \dots, a_n linearly independent, such that $\gamma(t)$, $t \in \mathbb{K}$, is a basis of

$\text{Ker}(d\alpha + td\beta)$ (see [11, 12].) Now choose $\rho \in \mathcal{A}^*$ such that $\rho(a_0) = 1$ and $\rho(a_k) = 0$, $k = 1, \dots, n$; then $\rho(\gamma(t)) \neq 0$ for every $t \in \mathbb{C}$ (in the real case complexify \mathcal{A} .)

On the other hand we identify $\mathcal{B}_{\mathcal{A}}^*$ and $\mathcal{A}^* \times \mathcal{A}^* \times (\mathbb{K})^*$ in the obvious way. Set $\Lambda_1 = \Lambda(0, \beta, 0)$; we shall prove that (Λ, Λ_1) is generic and non-flat at $(\rho, \alpha, 0)$.

First let us check that $\text{rank} \Lambda_1 = 2m$. As $\beta \wedge (d\beta)^{n-1} \neq 0$, there exists a basis $\{\tilde{e}_1, \dots, \tilde{e}_m\}$ of \mathcal{A} such that $\beta = \tilde{e}_1^*$ and $d\beta = -\sum_{j=1}^{n-1} \tilde{e}_{2j}^* \wedge \tilde{e}_{2j+1}^*$. Then in coordinates (x, y, z) of $\mathcal{B}_{\mathcal{A}}^*$ one has

$$\Lambda_1 = \sum_{j=1}^{n-1} \left(\frac{\partial}{\partial x_{2j}} \wedge \frac{\partial}{\partial y_{2j+1}} - \frac{\partial}{\partial x_{2j+1}} \wedge \frac{\partial}{\partial y_{2j}} \right) + \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial z}$$

whose rank equals $2m$.

Observe that dx_1 is a Casimir of Λ_1 , which corresponds to $e_1 \in \mathcal{B}_{\mathcal{A}}$. As $\text{rank}([e_1, \quad]) \geq 2$ because \mathcal{A} has trivial center, from lemmas 3 and 5 and remark 5 follows the non-flatness of (Λ, Λ_1) at $(\rho, \alpha, 0)$ provided that it is generic.

Since $\text{rank} \Lambda_1 = 2m$, for verifying that (Λ, Λ_1) is generic at $(\rho, \alpha, 0)$ it suffices to show that every $(\Lambda + t\Lambda_1)(\rho, \alpha, 0)$, $t \in \mathbb{C}$, has rank $2m$, which is equivalent to see that $\Lambda(\rho, \alpha + t\beta, 0)$, $t \in \mathbb{C}$, has rank $2m$. Set $\tau = \alpha + t\beta$; by complexifying \mathcal{A} if necessary, we may suppose that it is a complex Lie algebra without loss of generality. If $\tau \wedge (d\tau)^{n-1} \neq 0$ by considering a second basis $\{\tilde{e}_1, \dots, \tilde{e}_m\}$ of \mathcal{A} (denoted as the first one for sake of simplicity) such that $\tau = \tilde{e}_1^*$ and $d\tau = -\sum_{j=1}^{n-1} \tilde{e}_{2j}^* \wedge \tilde{e}_{2j+1}^*$, in coordinates (x, y, z) on $\mathcal{B}_{\mathcal{A}}^*$ one has

$$\begin{aligned} \Lambda(\rho, \tau, 0) &= \sum_{1 \leq i < j \leq m} a_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \\ &+ \sum_{j=1}^{n-1} \left(\frac{\partial}{\partial x_{2j}} \wedge \frac{\partial}{\partial y_{2j+1}} - \frac{\partial}{\partial x_{2j+1}} \wedge \frac{\partial}{\partial y_{2j}} \right) + \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial z} \\ &= \sum_{j=1}^{n-1} \left(\frac{\partial}{\partial x_{2j}} \wedge v_{2j+1} - \frac{\partial}{\partial x_{2j+1}} \wedge v_{2j} \right) + \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial z}, \end{aligned}$$

where $v_k = (\partial/\partial y_k) + \sum_{r=1}^m b_{kr}(\partial/\partial x_r)$, $k = 2, \dots, m$ for suitable scalars b_{kr} , whose rank equals $2m$.

Now assume $\tau \wedge (d\tau)^{n-1} = 0$. As $(d\tau)^{n-1} \neq 0$ since $(d\alpha, d\beta)$ is generic and $\rho|_{\text{Ker} d\tau} \neq 0$ because $\rho(\gamma(t)) \neq 0$, there exists a basis $\{\tilde{e}_1, \dots, \tilde{e}_m\}$ of \mathcal{A} (denoted as the first and second ones) such that $\tau = \tilde{e}_2^*$, $d\tau = -\sum_{j=1}^{n-1} \tilde{e}_{2j}^* \wedge \tilde{e}_{2j+1}^*$ and $\rho = \tilde{e}_1^*$. Thus $\{\tilde{e}_1, \tilde{e}_3\}$ is a basis of \mathcal{A}_τ , $[\tilde{e}_1, \tilde{e}_3] = c_{13}^1 \tilde{e}_1$ with $c_{13}^1 \neq 0$ since \mathcal{A}_τ is non-abelian, and in coordinates (x, y, z) on $\mathcal{B}_\mathcal{A}^*$ one has $x_1(\rho, \tau, 0) = y_2(\rho, \tau, 0) = 1$ while the remainder coordinates of $(\rho, \tau, 0)$ vanish. Therefore

$$\begin{aligned} \Lambda(\rho, \tau, 0) &= \sum_{1 \leq i < j \leq m} c_{ij}^1 \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \\ &\quad + \sum_{j=1}^{n-1} \left(\frac{\partial}{\partial x_{2j}} \wedge \frac{\partial}{\partial y_{2j+1}} - \frac{\partial}{\partial x_{2j+1}} \wedge \frac{\partial}{\partial y_{2j}} \right) + \frac{\partial}{\partial y_2} \wedge \frac{\partial}{\partial z} \\ &= c_{13}^1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_2} \wedge w_3 - \left(\frac{\partial}{\partial x_3} + \frac{\partial}{\partial z} \right) \wedge \frac{\partial}{\partial y_2} \\ &\quad + \sum_{j=2}^{n-1} \left(\frac{\partial}{\partial x_{2j}} \wedge w_{2j+1} - \frac{\partial}{\partial x_{2j+1}} \wedge w_{2j} \right), \end{aligned}$$

where $w_k = (\partial/\partial y_k) + \sum_{r=1}^m \tilde{b}_{kr}(\partial/\partial x_r)$, $k = 3, \dots, m$ for suitable scalars \tilde{b}_{kr} . Clearly its rank equals $2m$ since $c_{13}^1 \neq 0$. \square

Proposition 5. *Consider an unimodular Lie algebra \mathcal{A} of dimension $m = 2n - 1 \geq 3$; assume that this algebra possesses some contact form. If there exist $\alpha, \beta \in \mathcal{A}^*$ such that $(d\alpha, d\beta)$ is generic then on $\mathcal{B}_\mathcal{A}^*$ there exists some non-flat generic linear Poisson pair.*

Proof. The set of contact forms is open and dense in \mathcal{A}^* . As to be generic is an open property, there exists $\beta' \in \mathcal{A}^*$ such that $(d\alpha, d\beta')$ is generic and $\beta' \wedge (d\beta')^{n-1} \neq 0$. Now apply lemma 7 and proposition 4. \square

Example 4. Proposition 4 (proposition 5 is just a particular case of the foregoing one) can be applied to a 3-dimensional Lie algebra \mathcal{A} if

and only if either it is simple or there exists a basis $\{e_1, e_2, e_3\}$ such that $[e_1, e_2] = e_3$, $[e_1, e_3] = ae_2 + be_3$ with $a \neq 0$ and $[e_2, e_3] = 0$. Indeed, the simple case is obvious; therefore assume solvable \mathcal{A} .

If proposition 4 applies, then \mathcal{A} possess contact forms and its center is trivial (see the beginning of the proof of proposition 4.) In this case a computation shows the existence of this basis (as its center equals zero \mathcal{A} contains a 2-dimensional abelian ideal \mathcal{A}_0 such that $[v, \cdot]: \mathcal{A}_0 \rightarrow \mathcal{A}_0$ is an isomorphism for any $v \notin \mathcal{A}_0$; the existence of contact forms implies that $[v, \cdot]: \mathcal{A}_0 \rightarrow \mathcal{A}_0$ is never multiple of identity.)

Conversely, when a such basis exists it suffices to set $\alpha = e_2^*$ and $\beta = e_3^*$.

Example 5. Consider the Lie algebra \mathcal{A} of basis $\{e_1, \dots, e_{2n-1}\}$, $n \geq 2$, given by $[e_{2j-1}, e_{2j}] = -e_{2j}$ and $[e_{2j-1}, e_{2n-1}] = -ae_{2n-1}$ with $a \neq 0$, $j = 1, \dots, n-1$, and $[e_i, e_r] = 0$, $i < r$, otherwise (this algebra corresponds to consider vector fields $e_{2j-1} = \partial/\partial x_j$ and $e_{2j} = \exp(-x_j)(\partial/\partial x_n)$, $j = 1, \dots, n-1$, and $e_{2n-1} = \exp(-\sum_{k=1}^{n-1} ax_k)(\partial/\partial x_n)$ on \mathbb{K}^n .) Set $\alpha = \sum_{j=1}^{n-1} e_{2j}^*$ and $\beta = \sum_{j=1}^{n-1} a_j e_{2j}^* + e_{2n-1}^*$ where a_1, \dots, a_{n-1} are distinct and non-vanishing scalars.

Then

$$\begin{aligned} (d\alpha + td\beta)^{n-1} &= \left(\sum_{j=1}^{n-1} (a_j t + 1) e_{2j-1}^* \wedge e_{2j-1}^* + at \left(\sum_{j=1}^{n-1} e_{2j-1}^* \right) \wedge e_{2n-1}^* \right)^{n-1} \\ &= (n-1)! \left(\prod_{j=1}^{n-1} (a_j t + 1) \right) e_1^* \wedge \dots \wedge e_{2n-2}^* \\ &\quad + (n-1)! at \sum_{k=1}^{n-1} \left[\left(\prod_{j=1, j \neq k}^{n-1} (a_j t + 1) \right) e_1^* \wedge \dots \wedge \widehat{e_{2k}^*} \wedge \dots \wedge e_{2n-1}^* \right] \end{aligned}$$

so non-zero for any $t \in \mathbb{C}$.

On the other hand

$$(\alpha + t\beta) \wedge (d\alpha + td\beta)^{n-1} = (n-1)! (a[1-n] + 1)t \left(\prod_{j=1}^{n-1} (a_j t + 1) \right) e_1^* \wedge \dots \wedge e_{2n-1}^*$$

while $\beta \wedge (d\beta)^{n-1} = (n-1)!(a[1-n] + 1)a_1 \cdots a_{n-1}e_1^* \wedge \cdots \wedge e_{2n-1}^*$.

Let us suppose $a \neq (n-1)^{-1}$. Then β is a contact form and $(d\alpha, d\beta)$ is generic. Thus if $a = -1$, as \mathcal{A} is unimodular, one may apply proposition 5 to (α, β) . In the general case we have to examine algebras $\mathcal{A}_{(\alpha+t\beta)}$, $t \in \mathbb{C}$. They are of dimension two just when $t = 0, -a_1^{-1}, \dots, -a_{n-1}^{-1}$.

A basis of \mathcal{A}_α , which corresponds to $t = 0$, is $\{\sum_{j=1}^{n-1} e_{2j-1}, e_{2n-1}\}$, so this algebra is not abelian. Now suppose $t = -a_1^{-1}$ (the remainder cases are similar); then a basis of $\mathcal{A}_{(\alpha-a_1^{-1}\beta)}$ is $\{e_2, (a^{-1}-n+2)e_1 + \sum_{j=2}^{n-1} e_{2j-1}\}$ and it is non-abelian if and only if $a^{-1}-n+2 \neq 0$, that is $a \neq (n-2)^{-1}$.

Summing up, proposition 4 may be applied just when $a \neq 0, (n-1)^{-1}, (n-2)^{-1}$.

Example 6. Let \mathcal{A} be the truncated Lie algebra of dimension $m = 2n-1 \geq 3$ of example 2. Recall that the couple (de_m^*, de_{m-1}^*) was generic, so $(d\alpha, d\beta)$ where $\alpha = e_m^*$ and $\beta = e_m^* + e_{m-1}^*$ is generic too. By lemma 6 β is a contact form and $\mathcal{A}_{(\alpha+t\beta)} = \{0\}$ unless $t = 0, -1$. Therefore to see that (α, β) is generic one has to show that $\mathcal{A}_{e_m^*}$ and $\mathcal{A}_{e_{m-1}^*}$ are not abelian.

But $\{e_1, e_n\}$ is a basis of $\mathcal{A}_{e_m^*}$ and $[e_1, e_n] = (n-1)e_n$ while $\{e_1, e_m\}$ is a basis of $\mathcal{A}_{e_{m-1}^*}$ and $[e_1, e_m] = (m-1)e_m$, so (α, β) is generic and proposition 4 may be applied to it.

7. THE SPECIAL AFFINE ALGEBRA

In this section we show that proposition 5 may be applied to this Lie algebra. Let V be a real or complex vector space of dimension $n \geq 2$ and $\mathcal{A}ff(V)$ the affine algebra of V , which can be regarded too like the algebra of polynomial vector fields on V of degree ≤ 1 . Recall that $\mathcal{A}ff(V) = \mathcal{I} \oplus sl(V) \oplus V$ where \mathcal{I} consists of linear vector fields multiples of identity, $sl(V)$ is the special linear algebra of V and V the ideal of constant vector fields. On the other hand, the dual space $\mathcal{A}ff(V)^*$ will be identify to $\mathcal{I}^* \oplus sl(V)^* \oplus V^*$ in the obvious way.

Denote by kil the Killing form of $sl(V)$, which is non-degenerate; therefore if one sets $\alpha_g = kil(g, \cdot)$, $g \in sl(V)$, then $g \in sl(V) \rightarrow \alpha_g \in sl(V)^*$ is an isomorphism of vector spaces. Moreover, given $g, h \in sl(V)$ then $(d\alpha_g)(h, \cdot) = -\alpha_{[g,h]}$; thereby $(d\alpha_g)(h, \cdot) = 0$ if and only if $[g, h] = 0$.

For any $g \in gl(V)$ and $\tau \in V^*$ the subspace spanned by g, τ means that spanned by τ and the dual endomorphism g^* . One will need the following results:

Lemma 8. *On a real or complex vector space $E = E_1 \oplus E_2$, of even dimension, consider a couple of 2-forms λ, λ_1 such that $Ker \lambda \supset E_2$, $\lambda_1|_{E_1} = 0$ and $\lambda_1|_{E_2} = 0$. One has:*

- (a) *If $\lambda_1|_{Ker \lambda}$ is symplectic then $\lambda + \lambda_1$ is symplectic too.*
- (b) *If the corank of $\lambda_1|_{Ker \lambda}$ equals two and $\dim(E_1 \cap Ker(\lambda_1|_{Ker \lambda})) \geq 1$, then the corank of $\lambda + \lambda_1$ equals two.*

Lemma 9. *For any $n \geq 2$ one may find real numbers $a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n$ satisfying:*

- (I) *$a_i \neq a_j$, $b_i \neq b_j$ and $c_i \neq c_j$ whenever $i \neq j$; moreover no c_i , $i = 1, \dots, n$, vanishes.*
- (II) *$\sum_{i=1}^n a_i = \sum_{i=1}^n b_i = 0$.*
- (III) *For every $t \in \mathbb{C} - \{0\}$, at least $n - 1$ elements of the family $a_1 + tb_1, \dots, a_n + tb_n$ are different. Moreover is a such family includes two equal elements then $1 + tc_i \neq 0$ for every $i = 1, \dots, n$.*

Proof. Consider a family a'_1, \dots, a'_n of distinct rational numbers and a second one b'_1, \dots, b'_n of rationally independent real numbers. Set $t_{ij} = (a'_i - a'_j)(b'_i - b'_j)^{-1}$, $i \neq j$. Then $a'_i + tb'_i = a'_j + tb'_j$ if and only if $t = -t_{ij}$.

Suppose $a'_i + tb'_i = a'_j + tb'_j$ and $a'_k + tb'_k = a'_r + tb'_r$ for some $i < j$ and $k < r$ with $(i, j) \neq (k, r)$. Then $t = -t_{ij} = -t_{kr}$, that is $t_{ij} = t_{kr}$,

and an elementary computation shows that b'_1, \dots, b'_n are not rationally independent; therefore (III) holds for these two families.

Observe that (III) holds too for $a'_1 + a, \dots, a'_n + a$ and $b'_1 + b, \dots, b'_n + b$ whatever a, b are, because each t_{ij} does not change. In other words, setting $a = -\sum_{i=1}^n (a'_i/n)$ and $b = -\sum_{i=1}^n (b'_i/n)$ shows the existence of two families a_1, \dots, a_n and b_1, \dots, b_n satisfying (I), (II) and (III).

Finally, choose $c_1, \dots, c_n \in \mathbb{R} - (\{0\} \cup \{t_{ij}^{-1} \mid 1 \leq i < j \leq n\})$ that are distinct among them. \square

Proposition 6. *Given a n -dimensional, $n \geq 2$, real or complex vector space consider $g \in \mathfrak{sl}(V)$ and $\tau \in V^*$ and regard $\alpha_g + \tau$ like an element of $\mathcal{A}ff(V)^*$. Assume diagonalizable g . One has:*

(a) *If the eigenvalues of g are distinct and g, τ span V^* , then $d(\alpha_g + \tau)$ is symplectic.*

(b) *If, at least, $n - 1$ eigenvalues are different and g, τ span a vector subspace of dimension $n - 1$, then $\text{rank}(d(\alpha_g + \tau)) = n^2 + n - 2 = \dim \mathcal{A}ff(V) - 2$. Moreover $\text{Ker}(d(\alpha_g + \tau))$ is not included in $\mathfrak{sl}(V) \oplus V$.*

Proof. Set $E = \mathcal{A}ff(V)$, $E_1 = \mathcal{I} \oplus \mathfrak{sl}(V)$, $E_2 = V$, $\lambda = d\alpha_g$ and $\lambda_1 = d\tau$.

(a) In this case there is a basis $\{v_1, \dots, v_n\}$ of V such that $g = \sum_{j=1}^n a_j v_j \otimes v_j^*$, where $a_i \neq a_j$ if $i \neq j$, and $\tau = \sum_{j=1}^n v_j^*$. Then $\{v_1 \otimes v_1^*, \dots, v_n \otimes v_n^*, v_1, \dots, v_n\}$ is a basis of $\text{Ker} d\alpha_g$ and

$$d\tau|_{\text{Ker} d\alpha_g} = \pm \left(\sum_{j=1}^n (v_j \otimes v_j^*)^* \wedge v_j^* \right)_{| \text{Ker} d\alpha_g}$$

where $\{\{v_i \otimes v_j^*\}, i, j = 1, \dots, n, v_1, \dots, v_n\}$ is the basis of $\mathcal{A}ff(V)$ associated to $\{v_1, \dots, v_n\}$ and $\{\{(v_i \otimes v_j^*)^*\}, i, j = 1, \dots, n, v_1^*, \dots, v_n^*\}$ its dual basis. Now apply (a) of lemma 8.

(b) This time there are two possible cases. First assume that all eigenvalues of g are distinct; then there exists a basis $\{v_1, \dots, v_n\}$ of V such that $g = \sum_{j=1}^n a_j v_j \otimes v_j^*$, with $a_i \neq a_j$ if $i \neq j$, and $\tau = \sum_{j=1}^{n-1} v_j^*$.

On the other hand $\text{Ker}d\alpha_g$ is the same as before while

$$d\tau|_{\text{Ker}d\alpha_g} = \pm \left(\sum_{j=1}^{n-1} (v_j \otimes v_j^*)^* \wedge v_j^* \right)_{|_{\text{Ker}d\alpha_g}}$$

and it suffices applying (b) of lemma 8 for computing the rank.

Now suppose that two eigenvalues are equal; in this case there exists a basis $\{v_1, \dots, v_n\}$ of V such that $g = \sum_{j=1}^{n-2} a_j v_j \otimes v_j^* + a_{n-1}(v_{n-1} \otimes v_{n-1}^* + v_n \otimes v_n^*)$, with $a_i \neq a_j$ if $i \neq j$, and $\tau = \sum_{j=1}^{n-1} v_j^*$. Then $\{v_1 \otimes v_1^*, \dots, v_{n-2} \otimes v_{n-2}^*, \{v_k \otimes v_r^*\}, k, r = n-1, n, v_1, \dots, v_n\}$ is a basis of $\text{Ker}d\alpha_g$ and $d\tau|_{\text{Ker}d\alpha_g}$ equals

$$\pm \left(\sum_{j=1}^{n-2} (v_j \otimes v_j^*)^* \wedge v_j^* + (v_{n-1} \otimes v_{n-1}^*)^* \wedge v_{n-1}^* + (v_{n-1} \otimes v_n^*)^* \wedge v_n^* \right)_{|_{\text{Ker}d\alpha_g}}$$

and it is enough to apply (b) of lemma 8 for computing the rank.

Finally note that in both cases $v_n \otimes v_n^*$ belongs to $\text{Ker}(d\alpha_g + \tau)$. \square

Example 7. Let $\mathcal{A}ff_0(V)$ be the special affine algebra of V , that is $\mathcal{A}ff_0(V) = \mathfrak{sl}(V) \oplus V$. Consider a basis $\{v_1, \dots, v_n\}$ of V and scalars $a_1, \dots, a_n, b_1, \dots, b_n, c_1, \dots, c_n$ as in lemma 9. Set $g = \sum_{j=1}^n a_j v_j \otimes v_j^*$, $h = \sum_{j=1}^n b_j v_j \otimes v_j^*$, $\tau = \sum_{j=1}^n v_j^*$ and $\mu = \sum_{j=1}^n c_j v_j^*$. Let $\tilde{\alpha} = \alpha_g + \tau$ and $\tilde{\beta} = \alpha_h + \mu$ that are 1-forms on $\mathcal{A}ff(V)$; then $\alpha = \tilde{\alpha}|_{\mathcal{A}ff_0(V)}$ and $\beta = \tilde{\beta}|_{\mathcal{A}ff_0(V)}$ are contact forms on $\mathcal{A}ff_0(V)$. Indeed, we prove it for α the other case is analogous. By (a) of proposition 6 $d\tilde{\alpha}$ is symplectic, so there is $z \in \mathcal{A}ff(V)$ such that $i_z d\tilde{\alpha} = \tilde{\alpha}$, which implies that $L_z((d\tilde{\alpha})^{n(n+1)/2}) \neq 0$ that is to say $z \notin \mathcal{A}ff_0(V)$. But z is a basis of the kernel of $\tilde{\alpha} \wedge (d\tilde{\alpha})^{(n(n+1)/2)-1}$, hence its restriction to $\mathcal{A}ff_0(V)$, which equals $\alpha \wedge (d\alpha)^{(n(n+1)/2)-1}$, is a volume form.

Moreover $(d\alpha, d\beta)$ is generic. Let us see it. Clearly $\text{rank}(d\alpha)$ and $\text{rank}(d\beta)$ equal $\dim \mathcal{A}ff_0(V) - 1$, so it suffices to show that the rank of $d\alpha + td\beta$, $t \in \mathbb{C} - \{0\}$, is maximal. If $d(\tilde{\alpha} + t\tilde{\beta})$ is symplectic reason as before. If not, taking into account that $\tilde{\alpha} + t\tilde{\beta} = \alpha_{g+th} + (\tau + t\mu)$,

$g + th = \sum_{j=1}^n (a_j + tb_j)v_j \otimes v_j^*$ and $\tau + t\mu = \sum_{j=1}^n (1 + tc_j)v_j^*$, by proposition 6 and lemma 9 we have two cases:

- (1) The family $a_1 + tb_1, \dots, a_n + tb_n$ just includes two equal elements but no $1 + tc_i$, $i = 1, \dots, n$, vanishes.
- (2) All $a_1 + tb_1, \dots, a_n + tb_n$ are distinct but one element of the family $1 + tc_1, \dots, 1 + tc_n$ vanishes.

In both cases, by (b) of proposition 6, $\text{rank}(d(\tilde{\alpha} + t\tilde{\beta})) = n^2 + n - 2$ and $\text{Kerd}(\tilde{\alpha} + t\tilde{\beta}) \not\subset \mathcal{A}ff_0(V)$, so $\text{rank}(d(\alpha + t\beta)) = \text{rank}(d(\tilde{\alpha} + t\tilde{\beta})) = \dim \mathcal{A}ff_0(V) - 1$.

Summing up, one may apply proposition 5 to $\mathcal{A}ff_0(V)$ and (α, β) .

Example 8. Consider $\{v, \dots, v_n\}$, a_1, \dots, a_n , b_1, \dots, b_n , c_1, \dots, c_n as in the foregoing example. Let a be any scalar. Set $\mathcal{A}(V, a) = \mathcal{A}ff(V) \oplus \mathbb{K}$ endowed with the bracket defined below. Regard $\mathcal{A}ff(V)$, \mathbb{K} as subsets of $\mathcal{A}(V, a)$ and $\mathcal{A}ff(V)^*$, \mathbb{K}^* like subsets of $\mathcal{A}(V, a)^*$ in the obvious way. Let e be the unit of \mathbb{K} seen in $\mathcal{A}(V, a)$ and e^* the element of $\mathcal{A}(V, a)^*$ given by $e^*(e) = 1$, $e^*(\mathcal{A}ff(V)) = 0$. On the other hand let $id \in \mathcal{A}ff(V)$ be the morphism identity of V , that is $id = \sum_{j=1}^n v_j \otimes v_j^*$. Now we define a structure of Lie algebra on $\mathcal{A}(V, a)$, for which $\mathcal{A}ff(V)$ is a subalgebra, by putting $[id, e] = -ae$ and $[sl(V) \oplus V, e] = 0$.

Set $\alpha_1 = \tilde{\alpha}$ and $\beta_1 = \tilde{\beta} + e^*$ where $\tilde{\alpha}, \tilde{\beta}$, defined in the preceding example, are now regarded as elements of $\mathcal{A}(V, a)^*$. Note that $de^* = (a/n) \sum_{j=1}^n (v_j \otimes v_j^*)^* \wedge e^*$, so

$$\begin{aligned} (d\beta_1)^{n(n+1)/2} &= (d\tilde{\beta})^{n(n+1)/2} \\ &+ aC \left(\sum_{j=1}^n (v_j \otimes v_j^*)^* \right) \wedge (d\tilde{\beta})^{(n(n+1)/2)-1} \wedge e^* \neq 0 \end{aligned}$$

where C is non-zero constant, while

$$\beta_1 \wedge (d\beta_1)^{n(n+1)/2} = (aC' + 1)(d\tilde{\beta})^{n(n+1)/2} \wedge e^*$$

where C' is another constant (perhaps zero). As $d\tilde{\beta}$ is symplectic on $\mathcal{A}ff(V)$ it follows that β_1 is a contact form on $\mathcal{A}(V, a)$ if and only if $aC' + 1 \neq 0$.

On the other hand $(d\alpha_1, d\beta_1)$ is generic if $a \neq 0$. Indeed, since $d\tilde{\alpha}$ and $d\tilde{\beta}$ are symplectic on $\mathcal{A}ff(V)$ it is enough to show that $d\alpha_1 + td\beta_1$, $t \in \mathbb{C} - \{0\}$, has maximal rank. If $d\tilde{\alpha} + td\tilde{\beta}$ is symplectic on $\mathcal{A}ff(V)$ it is clear. Otherwise by proposition 6 the rank of $d\tilde{\alpha} + td\tilde{\beta}$ on $\mathcal{A}ff(V)$ equals $n^2 + n - 2$ and its kernel is not included in $sl(V) \oplus V$. But, always in $\mathcal{A}ff(V)$, $sl(V) \oplus V = \text{Ker}(\sum_{j=1}^n (v_j \otimes v_j^*)^*)$. Since $d\alpha_1 + td\beta_1$ equals $d\tilde{\alpha} + td\tilde{\beta}$ regarded on $\mathcal{A}(V, a)$ plus $(at/n) \sum_{j=1}^n (v_j \otimes v_j^*)^* \wedge e^*$, it follows that the rank of $d\alpha_1 + td\beta_1$ equals that of $d\tilde{\alpha} + td\tilde{\beta}$ plus two, that is $n^2 + n = \dim \mathcal{A}(V, a) - 1$.

Finally observe that $\mathcal{A}(V, a)$ is not unimodular if $a \neq n$. Thus one may choose a in such a way that β_1 is a contact form, $(d\alpha_1, d\beta_1)$ is generic and $\mathcal{A}(V, a)$ is not unimodular; in this case proposition 3 applied to $\mathcal{A}(V, a)$ and (α_1, β_1) shows the existence on $\mathcal{A}(V, a)^*$ of linear Poisson pairs which are generic and non-flat.

8. LIE POISSON PAIRS AND NIJENHUIS TORSION

In this section a method for constructing Lie Poisson pairs from an endomorphism with vanishing Nijenhuis is given. Recall that the Nijenhuis torsion of a $(1, 1)$ -tensor field J on a differentiable manifold is the $(1, 2)$ -tensor field N_J defined by $N_J(X, Y) = [JX, JY] + J^2[X, Y] - J[X, JY] - J[JX, Y]$.

Let \mathcal{A} be a Lie algebra and φ an endomorphism of \mathcal{A} as vector space; since φ can be seen like a left invariant $(1, 1)$ -tensor field on some Lie group, we may define its Nijenhuis torsion, which in linear terms is given by the formula $N_\varphi(a, b) = [\varphi a, \varphi b] + \varphi^2[a, b] - \varphi[a, \varphi b] - \varphi[\varphi a, b]$.

Set $[a, b]_1 = [a, \varphi b] + [\varphi a, b] - \varphi[a, b]$, $a, b \in \mathcal{A}$; then $[a, b] + t[a, b]_1 = [a, (I + t\varphi)b] + [(I + t\varphi)a, b] - (I + t\varphi)[a, b]$. Now assume $N_\varphi = 0$; if $I + t\varphi$

is invertible then $(I + t\varphi)^{-1}[(I + t\varphi)a, (I + t\varphi)b] = [a, (I + t\varphi)b] + [(I + t\varphi)a, b] - (I + t\varphi)[a, b]$ since $N_{(I+t\varphi)} = 0$. Therefore $[\cdot, \cdot] + t[\cdot, \cdot]_1$ defines a structure of Lie algebra and $(I + t\varphi): (\mathcal{A}, [\cdot, \cdot] + t[\cdot, \cdot]_1) \rightarrow (\mathcal{A}, [\cdot, \cdot])$ is an isomorphism of Lie algebras. As $(I + t\varphi)$ is invertible for almost every $t \in \mathbb{K}$, it follows that $N_\varphi = 0$ implies that $[\cdot, \cdot]_1$ is a Lie bracket which is compatible with $[\cdot, \cdot]$. In this case the couple of Lie-Poisson structures (Λ, Λ_1) , associated to $[\cdot, \cdot]$ and $[\cdot, \cdot]_1$ respectively, is a Lie Poisson pair. Moreover $(I + t\varphi)^*: (\mathcal{A}^*, \Lambda) \rightarrow (\mathcal{A}^*, \Lambda + t\Lambda_1)$ is a Poisson diffeomorphism whenever $(I + t\varphi)$ is invertible.

Example 9. Consider the truncated Lie algebra \mathcal{A} of dimension $m = 2n - 1 \geq 5$ and the basis $\{e_1, \dots, e_m\}$ given in example 2. Let $\varphi = e_n \otimes e_m^*$; then $N_\varphi = 0$. Besides the associated Lie algebra $(\mathcal{A}, [\cdot, \cdot]_1)$ is unimodular and e_n is a basis of its center.

In coordinates (x_1, \dots, x_m) with respect to the dual basis $\{e_1^*, \dots, e_m^*\}$ one has:

$$\begin{aligned} \Lambda_1 = & x_n \sum_{i=2}^{n-1} 2(i-n) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_{2n-i}} \\ & + \left((1-n)x_n \frac{\partial}{\partial x_1} + \sum_{i=2}^{n-1} (n-i)x_{n+i-1} \frac{\partial}{\partial x_i} \right) \wedge \frac{\partial}{\partial x_{2n-1}} \end{aligned}$$

while

$$\Lambda_1^{n-1} = C x_n^{n-1} \frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_{n-1}} \wedge \frac{\partial}{\partial x_{n+1}} \wedge \dots \wedge \frac{\partial}{\partial x_{2n-1}}$$

where C is a non-vanishing constant. Thus $\text{rank} \Lambda_1 = 2n - 2$ if $x_n \neq 0$. Observe that dx_n is a Casimir of Λ_1 .

On the other hand

$$\begin{aligned} \Lambda = & \sum_{1 \leq i < j \leq 2n-1; i+j \leq 2n} (j-i)x_{i+j-1} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \\ = & \frac{\partial}{\partial x_1} \wedge \left(x_2 \frac{\partial}{\partial x_2} + \dots + (2n-2)x_{2n-1} \frac{\partial}{\partial x_{2n-1}} \right) \\ & + \frac{\partial}{\partial x_2} \wedge \left(x_4 \frac{\partial}{\partial x_3} + \dots + (2n-4)x_{2n-1} \frac{\partial}{\partial x_{2n-2}} \right) \end{aligned}$$

$$+ \cdots + \frac{\partial}{\partial x_{n-1}} \wedge \left(x_{2n-2} \frac{\partial}{\partial x_n} + 2x_{2n-1} \frac{\partial}{\partial x_{n+1}} \right)$$

Therefore $\Lambda^{n-1}(x) \neq 0$ if and only if the vector fields given by the parentheses are linearly independent modulo $(\partial/\partial x_1), \dots, (\partial/\partial x_{n-1})$, which just happens when, at least, $x_{2n-2} \neq 0$ or $x_{2n-1} \neq 0$.

Let $A = \{x \in \mathcal{A}^* \mid x_n \neq 0, x_{2n-2} \neq 0\}$. Since φ is nilpotent $(I+t\varphi)$ is always invertible; moreover $(I+t\varphi)^*(x) = (x_1, \dots, x_{2n-2}, x_{2n-1} + tx_n)$, so $(I+t\varphi)^*(A) = A$, which implies that $(\Lambda + t\Lambda_1)^{n-1}(x) \neq 0$, $t \in \mathbb{K}$, $x \in A$, since $(I+t\varphi)^*$ transforms Λ in $\Lambda + t\Lambda_1$. Therefore (Λ, Λ_1) is generic on A . Finally by lemmas 3 and 5 and remark 5 (first paragraph) the Lie Poisson pair (Λ, Λ_1) is not flat at any point of A , because dx_n is a Casimir of Λ_1 and $\text{rank}([e_n, \quad]) \geq 2$.

Proposition 7. *Let \mathcal{A} be a non-unimodular Lie algebra of dimension $m = 2n - 1 \geq 3$. If there exist $\alpha, \beta \in \mathcal{A}^*$ such that β is a contact form and $(d\alpha, d\beta)$ is generic, then on the dual space of the product Lie algebra $\mathcal{A} \times \mathcal{Aff}(\mathbb{K})$ there exists some generic and non-flat Lie Poisson pair.*

Proof. Consider a basis $\{\tilde{e}_1, \dots, \tilde{e}_m\}$ of \mathcal{A} and an another one $\{\tilde{f}_1, \tilde{f}_2\}$ of $\mathcal{Aff}(\mathbb{K})$ such that $\beta = \tilde{e}_m^*$, $d\beta = \sum_{i=1}^{n-1} \tilde{e}_{2i-1}^* \wedge \tilde{e}_{2i}^*$ and $[\tilde{f}_1, \tilde{f}_2] = \tilde{f}_1$. Set $e_i = (\tilde{e}_i, 0)$, $i = 1, \dots, m$, $f_j = (0, \tilde{f}_j)$, $j = 1, 2$, $\alpha_1 = \alpha + f_1^*$ and $\beta_1 = \beta$ where α, β are regarded this time like 1-forms on $\mathcal{A} \times \mathcal{Aff}(\mathbb{K})$ in the obvious way (that is $\alpha(\{0\} \times \mathcal{Aff}(\mathbb{K})) = \beta(\{0\} \times \mathcal{Aff}(\mathbb{K})) = 0$.) Then $\{e_1, \dots, e_m, f_1, f_2\}$ is a basis of $\mathcal{A} \times \mathcal{Aff}(\mathbb{K})$; let $\{e_1^*, \dots, e_m^*, f_1^*, f_2^*\}$ be its dual basis and $(x_1, \dots, x_m, y_1, y_2)$ the associated coordinates on $(\mathcal{A} \times \mathcal{Aff}(\mathbb{K}))^*$

It is easily checked that the Nijenhuis torsion of $f_1 \otimes \beta_1$ vanishes, so we have a second Lie bracket $[\quad, \quad]_1$, which is compatible with the product bracket $[\quad, \quad]$. Let d, d_1 the respective exterior derivatives and Λ, Λ_1 the Lie-Poisson structures on $(\mathcal{A} \times \mathcal{Aff}(\mathbb{K}))^*$. We shall show that the Lie Poisson pair (Λ, Λ_1) is generic and non-flat.

First one checks the genericity of $(\Lambda, \Lambda_1)(\alpha_1)$, which is equivalent to that of $(d\alpha_1, d_1\alpha_1)$. As $d\alpha_1 = d\alpha - f_1^* \wedge f_2^*$ and $d_1\alpha_1 = -d\beta - e_m^* \wedge f_2^*$, where $d\alpha$ and $d\beta$ are computed on \mathcal{A} and then extended to $\mathcal{A} \times \mathcal{A}ff(\mathbb{K})$ in the natural way, the rank of both 2-forms equals $2n$. Therefore it will suffice to show that $\text{rank}(d\alpha_1 + td_1\alpha_1) = 2n$, $t \in \mathbb{C} - \{0\}$, which is obvious since $d\alpha_1 + td_1\alpha_1 = (d\alpha - td\beta) - (f_1^* + te_m^*) \wedge f_2^*$ and $(d\alpha, d\beta)$ is generic on \mathcal{A} .

Let $(\omega, \omega_1, \Omega)$ be a representative of (Λ, Λ_1) . As $(\mathcal{A} \times \mathcal{A}ff(\mathbb{K}), [\cdot, \cdot]_1)$ is unimodular $d\omega_1 = 0$; on the other hand dy_1 is a Casimir of Λ_1 since f_1 belongs to the center of this algebra. By lemma 5 the vector field $X = \sum_{j=1}^m (\text{tr}[e_j, \cdot])(\partial/\partial x_j) + \sum_{k=1}^2 (\text{tr}[f_k, \cdot])(\partial/\partial y_k)$ is a basis of $\text{Ker}d\omega$; moreover since \mathcal{A} is non-unimodular some $(\text{tr}[e_j, \cdot])(\partial/\partial x_j)$ does not vanish, so $X \wedge (\partial/\partial y_1) \neq 0$. Assume that (Λ, Λ_1) is flat at α_1 ; then, by lemma 3, X and $\Lambda(dy_1, \cdot) = y_1(\partial/\partial y_1)$ have to be proportional, so $X \wedge (\partial/\partial y_1) = 0$ *contradiction*. \square

Remark 6. The hypotheses of propositions 4 and 7 are rather close and almost the same examples illustrate both results. Thus proposition 7 may be applied to example 4 when $[v, \cdot]: \mathcal{A}_0 \rightarrow \mathcal{A}_0$ has non-vanishing trace, to example 5 if $a \neq -1, 0, (n-1)^{-1}, (n-2)^{-1}$, always to example 6 and, finally, to example 8 when a is chosen in such a way that $\mathcal{A}(V, a)$ is non-unimodular, β_1 is a contact form and $(d\alpha_1, d\beta_1)$ generic.

Of course example 7 has to be excluded since its Lie algebra is unimodular.

9. GENERIC NON-FLAT LINEAR POISSON PAIRS IN DIMENSION 3

The purpose of this section and the next one is to illustrate proposition 2. Of course even if the our approach is new, as dimension three has been well studied, most of the results of both sections are known, perhaps stated in a different way. Let \mathcal{A} be a 3-dimensional

non-unimodular Lie algebra and I_0 its unimodular ideal, whose dimension equals two (recall that unimodular implies flatness.) As Lie algebra I_0 itself is unimodular so abelian; therefore if $u, v \notin I_0$ then $[v, \cdot] = s[u, \cdot]$ for some $s \in \mathbb{K} - \{0\}$. Thus, up to non-vanishing multiplicative constant, one obtain a well-determined endomorphism $[u, \cdot]_{|I_0} \neq 0$.

Proposition 8. *Consider a 3-dimensional non-unimodular Lie algebra \mathcal{A} . Then on \mathcal{A}^* there exists some generic linear Poisson pair that is non-flat if and only if the endomorphism $[u, \cdot]_{|I_0}$, $u \notin I_0$, is not a multiple of identity.*

Let us see that. Consider a basis $\{e_1, e_2, e_3\}$ of \mathcal{A} such that $\{e_2, e_3\}$ is a basis of I_0 . Set $[e_1, e_2] = a_{22}e_2 + a_{23}e_3$ and $[e_1, e_3] = a_{32}e_2 + a_{33}e_3$; note that $a_{22} + a_{33} \neq 0$ since \mathcal{A} is non-unimodular. Moreover $\{e_1^* \wedge e_2^*, e_1^* \wedge e_3^*\}$ is a basis of the vector space of 2-cocycles. In coordinates (x_1, x_2, x_3) associates to the dual basis of $\{e_1, e_2, e_3\}$ one has

$$\Lambda = (\partial/\partial x_1) \wedge [(a_{22}x_2 + a_{23}x_3)(\partial/\partial x_2) + (a_{32}x_2 + a_{33}x_3)(\partial/\partial x_3)],$$

so $\omega = -(a_{32}x_2 + a_{33}x_3)dx_2 + (a_{22}x_2 + a_{23}x_3)dx_3$ and $\Omega = dx_1 \wedge dx_2 \wedge dx_3$ represent Λ .

In turn, any constant and Λ -compatible Poisson structure Λ_1 writes $\Lambda_1 = (\partial/\partial x_1) \wedge [b_2(\partial/\partial x_2) + b_3(\partial/\partial x_3)]$, $b_2, b_3 \in \mathbb{K}$, and $\omega_1 = -b_3dx_2 + b_2dx_3$, Ω represent it (we do not specify the dependence on (b_2, b_3) of Λ_1 .)

On the other hand $(\Lambda, \Lambda_1)(x)$ is generic if and only if $(a_{22}b_3 - a_{32}b_2)x_2 + (a_{23}b_3 - a_{33}b_2)x_3 \neq 0$; therefore the set of $(b_2, b_3) \in \mathbb{K}^2$ such that (Λ, Λ_1) has no generic point is always included in a vector line of \mathbb{K}^2 (given for example by $a_{22}b_3 - a_{32}b_2 = 0$ or by $a_{23}b_3 - a_{33}b_2 = 0$; at least one of these equations is not trivial since some a_{ij} does not vanish.)

As (Λ, Λ_1) is compatible and $d\omega_1 = 0$, the 1-form λ given by proposition 2 is a functional multiple of ω_1 . Now a computation shows that

$$\lambda = (a_{22} + a_{33})[(a_{32}b_2 - a_{22}b_3)x_2 + (a_{33}b_2 - a_{23}b_3)x_3]^{-1}(-b_3dx_2 + b_2dx_3).$$

Note that λ is just defined at any generic point of (Λ, Λ_1) . It is easily seen that $d\lambda = 0$ if and only if

$$a_{32}b_2^2 + (a_{33} - a_{22})b_2b_3 - a_{23}b_3^2 = 0.$$

When $[u, \cdot]_{|I_0}$, $u \notin I_0$ is a multiple of identity, automatically $d\lambda = 0$ and (Λ, Λ_1) is flat. Otherwise $a_{32}b_2^2 + (a_{33} - a_{22})b_2b_3 - a_{23}b_3^2$ can be regarded as a non-trivial quadratic form in (b_2, b_3) , which allows to choose $(b_2, b_3) \in \mathbb{K}^2$ in such a way that the set of generic points of (Λ, Λ_1) is not empty and $a_{32}b_2^2 + (a_{33} - a_{22})b_2b_3 - a_{23}b_3^2 \neq 0$, so $d\lambda \neq 0$ and (Λ, Λ_1) is non-flat.

10. GENERIC NON-FLAT LIE POISSON PAIRS IN DIMENSION 3

This time consider a 3-dimensional real or complex vector space \mathcal{A} endowed with two compatible Lie brackets $[\cdot, \cdot]$, $[\cdot, \cdot]_1$, and their respective Lie-Poisson structures Λ, Λ_1 on \mathcal{A}^* . One wants to describe when (Λ, Λ_1) is generic and non-flat, therefore all flat cases will be put aside. Observe that at least one of the bracket, for example $[\cdot, \cdot]$, has to be non-unimodular, otherwise (Λ, Λ_1) is flat. Now replacing $[\cdot, \cdot]_1$ by $[\cdot, \cdot]_1 + s[\cdot, \cdot]$ for a suitable scalar s if necessary, allows to suppose non-unimodular $[\cdot, \cdot]_1$ as well. Let I be the unimodular ideal of $[\cdot, \cdot]$ and I_1 that of $[\cdot, \cdot]_1$; then $(I, [\cdot, \cdot])$ and $(I_1, [\cdot, \cdot]_1)$ are abelian and 2-dimensional (see the foregoing section.) Moreover $I = I_1$.

Indeed, if $I \neq I_1$ then $\mathcal{A} = I + I_1$ and there exists a basis $\{e_1, e_2, e_3\}$ of \mathcal{A} such that $\{e_1, e_2\}$ is a basis of I and $\{e_2, e_3\}$ of I_1 . Let (x_1, x_2, x_3) be the coordinates of \mathcal{A}^* relative to the dual basis. With respect to $\Omega = dx_1 \wedge dx_2 \wedge dx_3$ the Lie-Poisson structures Λ and Λ_1 are represented by $\omega = (a_{11}x_1 + a_{12}x_2)dx_1 + (a_{21}x_1 + a_{22}x_2)dx_2$ and $\omega_1 = (b_{22}x_2 + b_{23}x_3)dx_2 + (b_{32}x_2 + b_{33}x_3)dx_3$ respectively, where a_{ij}, b_{kr} are scalars (see section 9 again.)

As $d\omega = adx_1 \wedge dx_2 \neq 0$ and $d\omega_1 = bdx_2 \wedge dx_3 \neq 0$ because $[\cdot, \cdot]$ and $[\cdot, \cdot]_1$ are non-unimodular, the 1-form λ given by proposition 2

necessarily equals fdx_2 for some function f . But $fdx_2 \wedge \omega = d\omega = adx_1 \wedge dx_2$ is closed, so $f = f(x_1, x_2)$. On the other hand, reasoning with ω_1 as before shows that $f = f(x_2, x_3)$. Therefore $f = f(x_2)$; but in this case $\lambda = f(x_2)dx_2$ is closed and (Λ, Λ_1) flat.

In other words, there exists a 2-dimensional vector subspace I of \mathcal{A} which is the unimodular ideal of both brackets. Therefore given any $u \notin I$ the structure is determined by the restriction of $[u, \cdot]$ and $[u, \cdot]_1$ to I . These two endomorphisms of I have non-vanishing trace (obviously the trace before and after restriction to I is the same), which shows the existence of an unimodular bracket $s[\cdot, \cdot] + s_1[\cdot, \cdot]_1$, for some $s, s_1 \in \mathbb{K} - \{0\}$. Note that up to non-zero multiplicative constant this bracket is unique. Now replacing $[\cdot, \cdot]_1$ by $s[\cdot, \cdot] + s_1[\cdot, \cdot]_1$ and calling it $[\cdot, \cdot]_1$ again, allows to suppose unimodular $[\cdot, \cdot]_1$ (of course one may consider $t[\cdot, \cdot] + t_1[\cdot, \cdot]_1$, $t \in \mathbb{K} - \{0\}$ and $t_1 \in \mathbb{K}$, instead $[\cdot, \cdot]$ if desired.)

In short, up to linear combinations of brackets, our problem is reduced to consider two Lie brackets $[\cdot, \cdot]$ and $[\cdot, \cdot]_1$, non-unimodular the first one and unimodular but non-zero the second one, such that the unimodular ideal I of $[\cdot, \cdot]$ is, at the same time, an abelian ideal of $[\cdot, \cdot]_1$. Note that $[\cdot, \cdot]$, $[\cdot, \cdot]_1$ automatically are compatible.

Observe that the endomorphism $[u, \cdot]_{1|I}$, $u \notin I$, is unique up to non-zero multiplicative constant, while $[u, \cdot]_I$ is determined up to non-vanishing multiplicative constant plus any multiple of $[u, \cdot]_{1|I}$. Thus the existence of some eigenvector of $[u, \cdot]_{1|I}$ (perhaps with complex eigenvalue in the real case) which is not eigenvector of $[u, \cdot]_I$ is independent of the choice of u , $[\cdot, \cdot]$ and $[\cdot, \cdot]_1$; in other words it is an intrinsic property of the Lie Poisson pair (more exactly of the Lie Poisson pencil.)

Proposition 9. *The foregoing Lie Poisson pair (Λ, Λ_1) , on \mathcal{A}^* , is generic and non-flat if and only if there exists some eigenvector of $[u, \cdot]_{1|I}$ that is not an eigenvector of $[u, \cdot]_I$.*

Proof. Consider a basis $\{e_1, e_2, e_3\}$ of \mathcal{A} such that $\{e_2, e_3\}$ is a basis of I . As $\text{tr}([e_1, \cdot]_{1|I}) = 0$ but $[e_1, \cdot]_{1|I} \neq 0$, the vector space I is cyclic and one can choose $\{e_2, e_3\}$ in such a way that $[e_1, e_2]_1 = e_3$ and $[e_1, e_3]_1 = be_2$. Set $[e_1, e_2] = a_{22}e_2 + a_{23}e_3$, $[e_1, e_3] = a_{32}e_2 + a_{33}e_3$. In coordinates (x_1, x_2, x_3) on \mathcal{A}^* associated to the dual basis, and with respect to $\Omega = dx_1 \wedge dx_2 \wedge dx_3$, the Poisson structures Λ and Λ_1 are respectively represented by $\omega = -(a_{32}x_2 + a_{33}x_3)dx_2 + (a_{22}x_2 + a_{23}x_3)dx_3$ and $\omega_1 = -bx_2dx_2 + x_3dx_3$.

Note that $(\Lambda, \Lambda_1)(x)$ is generic just when $(\omega \wedge \omega_1)(x) \neq 0$, that is to say just when $P(x) \neq 0$ where $P = a_{22}bx_2^2 + (a_{23}b - a_{32})x_2x_3 - a_{33}x_3^2$. Therefore (Λ, Λ_1) is generic if and only if P is not identically zero.

As $d\omega_1 = 0$ the 1-form λ of proposition 2 equals $f\omega_1$ for some function f . On the other hand $d\omega = (a_{22} + a_{33})dx_2 \wedge dx_3 = \lambda \wedge \omega = f\omega_1 \wedge \omega$ and a computation shows that $f = -(a_{22} + a_{33})P^{-1}$. Thus (Λ, Λ_1) will be flat if and only if $\lambda = -(a_{22} + a_{33})P^{-1}\omega_1$ is closed. Since $\omega_1 = -(1/2)dQ$ where $Q = bx_2^2 - x_3^2$, this is equivalent to say that $dP \wedge dQ = 0$; that is to say, if and only if P and Q are proportional as polynomials. Since Q does not identically vanish, the foregoing condition is equivalent to the existence of $c \in \mathbb{K}$ such that $(a_{22}b, a_{23}b - a_{32}, -a_{33}) = c(b, 0, -1)$.

First suppose $b = 0$. Then c does not exist just when $a_{32} \neq 0$, that is just when e_3 , which determines the single eigendirection of $[e_1, \cdot]_{1|I}$, is not an eigenvector of $[e_1, \cdot]_I$. Observe that $a_{32} \neq 0$ implies $P \neq 0$.

Now assume $b \neq 0$. Then $P \neq 0$ since $a_{22} + a_{33} \neq 0$. By replacing $\begin{bmatrix} e_2 \\ e_3 \end{bmatrix}$ by $\begin{bmatrix} e_2 \\ e_3 \end{bmatrix} - a_{23}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}_1$ if necessary, one may suppose $a_{23} = 0$ without loss of generality; in this case e_2 is an eigenvector of $[e_1, \cdot]_{1|I}$. On the other hand $(a_{22}b, -a_{32}, -a_{33}) = c(b, 0, -1)$ if and only if $a_{32} = 0$ and $a_{22} = a_{33}$; that is if and only if $[e_1, \cdot]_{1|I}$ is multiple of identity.

Consequently any eigenvector of $[e_1, \cdot]_{1|I}$ is eigenvector of $[e_1, \cdot]_I$ when c exists.

Conversely, if any eigenvector of $[e_1, \cdot]_{1|I}$ is eigenvector of $[e_1, \cdot]_I$ then, as $b \neq 0$, there exist two eigendirections of $[e_1, \cdot]_I$, coming from $[e_1, \cdot]_{1|I}$, which are different from the direction associated to e_2 ; therefore $[e_1, \cdot]_I$ has three distinct eigendirections and necessarily is multiple of identity. \square

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(F.J. Turiel) GEOMETRÍA Y TOPOLOGÍA, FACULTAD DE CIENCIAS, CAMPUS DE
TEATINOS, S/N, 29071-MÁLAGA, SPAIN
E-mail address: `turiel@uma.es`